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DYNAMIC THEORY OF QUASILINEAR PARABOLIC EQUATIONS II. REACTION-DIFFUSION SYSTEMS

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Introduction. Let Ω be a bounded smooth domain in $\mathbb{R}^n$ and consider a second order differential equation of the form

$$\partial_t u - \partial_j(a_{jk}(\cdot, u)\partial_k u) = f(\cdot, u, \partial u) \quad \text{on} \quad \Omega \times (0, \infty) \tag{1}$$

acting on $\mathbb{R}^N$-valued functions $u = (u^1, \ldots, u^N)$. (We use the summation convention throughout, j and k running from 1 to n, and r and s running from 1 to N.) We assume that

$$a_{jk} \in C^\infty(\overline{\Omega} \times G, \mathcal{L}(\mathbb{R}^N)), \quad 1 \le j, k \le n,$$

where G is an open subset of $\mathbb{R}^N$ and $\mathcal{L}(\mathbb{R}^N)$ is the space of all real $N \times N$ matrices. We assume also that

$$f \in C^\infty(\overline{\Omega} \times G \times \mathbb{R}^{nN}, \mathbb{R}^N)$$

and that f is 'affine in the gradient', that is,

$$f(\cdot, \cdot, \eta) = f_0 + f_j \eta_j, \quad \eta := (\eta_1, \ldots, \eta_n) \in \mathbb{R}^N \times \ldots \times \mathbb{R}^N,$$

where $f_0 : \overline{\Omega} \times G \to \mathbb{R}^N$ and $f_j : \overline{\Omega} \times G \to \mathcal{L}(\mathbb{R}^N), 1 \le j \le n$.

Equation (1) has to be complemented by boundary conditions, which are typically 'Dirichlet boundary conditions',

$$u = 0 \quad \text{on} \quad \partial\Omega \times (0, \infty), \tag{2}$$

or 'Neumann type boundary conditions',

$$a_{jk}(\cdot, u)\nu^j \partial_k u = g(\cdot, u) \quad \text{on} \quad \partial\Omega \times (0, \infty), \tag{3}$$

where $\nu := (\nu^1, \ldots, \nu^n)$ is the outer unit normal vector field on $\partial\Omega$ and

$$g \in C^\infty(\partial\Omega \times G, \mathbb{R}^N).$$

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If we assume that G is star shaped with respect to 0 and $g(\cdot, 0) = 0$ we can rewrite (3) in the form

$$a_{jk}(\cdot, u)\nu^j \partial_k u + b_0(\cdot, u)u = 0 \quad \text{on} \quad \partial\Omega \times (0, \infty), \tag{4}$$

where

$$b_0 \in C^\infty(\partial\Omega \times G, \mathcal{L}(\mathbb{R}^N))$$

is given by

$$b_0(\cdot, \xi) := -\int_0^1 \partial_2 g(\cdot, t\xi)\, dt, \qquad \xi \in G.$$

Of course, there will also occur combinations of (2) and (3), namely Dirichlet boundary conditions for some components of u and Neumann type boundary conditions for the remaining components. Moreover this configuration may change from component to component of $\partial\Omega$. This can be expressed concisely by requiring that

$$\mathcal{B}(u)u := \delta(a_{jk}(\cdot, u)\nu^j \partial_k u + b_0(\cdot, u)u) + (1 - \delta)u = 0 \quad \text{on} \quad \partial\Omega \times (0, \infty), \tag{5}$$

where $\delta := \operatorname{diag}[\delta^1, \dots, \delta^N]$ with $\delta^r \in C(\partial\Omega, \{0, 1\})$ for $1 \le r \le N$. Thus, if we put

$$\mathcal{A}(u)u := -\partial_j(a_{jk}(\cdot, u)\partial_k u),$$

we can write the systems (1), (5) in the concise form

$$\begin{aligned} \partial_t u + \mathcal{A}(u)u &= f(\cdot, u, \partial u) && \text{on } \Omega \times (0, \infty), \\ \mathcal{B}(u)u &= 0 && \text{on } \partial\Omega \times (0, \infty). \end{aligned} \tag{6}$$

In many concrete problems these equations are of a more special type. Namely they are in 'separated divergence form,' that is,

$$a_{jk}(\cdot, u) = \mathbf{A}(\cdot, u)\alpha_{jk}$$

where

$$\mathbf{A} \in C^\infty(\overline{\Omega} \times G, \mathcal{L}(\mathbb{R}^N))$$

and

$$\alpha := [\alpha_{jk}] \in C^\infty(\overline{\Omega}, \mathcal{L}(\mathbb{R}^n))$$

is symmetric and uniformly positive definite. In this case the boundary operator $\mathcal{B}(u)$ has the form

$$\mathcal{B}(u)u = \delta(\mathbf{A}(\cdot, u)\partial_{\nu_\alpha} u + b_0(\cdot, u)u) + (1 - \delta)u,$$

where ∂_{ν_α} is the conormal derivative with respect to the matrix α.

It is useful to consider a specific example. In certain population dynamical models there occur systems of the form

$$\begin{aligned} \partial_t v - \Delta[(\alpha_1 + \beta_{11} v + \beta_{12} w)v] - \operatorname{div}(\gamma_1 v \operatorname{grad} \Phi) &= v h_1(\cdot, v, w), \\ \partial_t w - \Delta[(\alpha_2 + \beta_{21} v + \beta_{22} w)w] - \operatorname{div}(\gamma_2 w \operatorname{grad} \Phi) &= w h_2(\cdot, v, w), \end{aligned} \tag{7}$$

where $\alpha_1, \alpha_2, \beta_{11}, \dots, \beta_{22}, \gamma_1, \gamma_2 \in C^\infty(\overline{\Omega}, \mathbb{R}^+)$, α_1 and α_2 are everywhere strictly positive, $\Phi \in C^\infty(\overline{\Omega}, \mathbb{R})$, and $h_j \in C^\infty(\overline{\Omega} \times \mathbb{R}^2, \mathbb{R})$, $j = 1, 2$. In this connection nonnegative solutions ($v \geq 0, w \geq 0$) are of interest only (cf. [38, 31]). Setting

$$J_1 := \operatorname{grad}[(\alpha_1 + \beta_{11}v + \beta_{12}w)v] + \gamma_1 v \operatorname{grad}\Phi,$$

$$J_2 := \operatorname{grad}[(\alpha_2 + \beta_{21}v + \beta_{22}w)w] + \gamma_2 w \operatorname{grad}\Phi,$$

the boundary conditions which are actually being considered in [38] are

$$(J_1|\nu) = 0, \quad (J_2|\nu) = 0 \quad \text{on } \partial\Omega \times (0, \infty). \tag{8}$$

This system can be written in the form (6) and is then of separated divergence form, where $\alpha_{jk} := \delta_{jk}$ (the Kronecker symbol),

$$\mathbf{A}(\cdot, \xi) := \begin{bmatrix} \alpha_1 + 2\beta_{11}\xi^1 + \beta_{12}\xi^2 & \beta_{12}\xi^1 \\ \beta_{21}\xi^2 & \alpha_2 + \beta_{21}\xi^1 + 2\beta_{22}\xi^2 \end{bmatrix}, \tag{9}$$

$$b_0(\cdot, \xi) := \operatorname{diag}[\partial_\nu\alpha_1 + \partial_\nu\beta_{11}\xi^1 + \partial_\nu\beta_{12}\xi^2 + \gamma_1\partial_\nu\Phi,\ \partial_\nu\alpha_2 + \partial_\nu\beta_{21}\xi^1 + \partial_\nu\beta_{22}\xi^2 + \gamma_2\partial_\nu\Phi]$$

for $\xi := (\xi^1, \xi^2) \in \mathbb{R}^2$, where $\delta := 1$, and where f is 'affine in the gradient'.

In order to obtain a powerful theory we shall have to impose an ellipticity condition upon 'the boundary value problems' $(\mathcal{A}(u), \mathcal{B}(u))$. To discuss this point we consider first a 'linear principal part system' $(\mathcal{A}_\pi, \mathcal{B}_\pi)$, where

$$\mathcal{A}_\pi u := -\partial_j(a_{jk}\partial_k u), \quad \mathcal{B}_\pi u := \delta a_{jk}\nu^j\partial_k u + (1 - \delta)u$$

and $a_{jk} \in C^\infty(\overline{\Omega}, \mathcal{L}(\mathbb{R}^N))$, $1 \leq j, k \leq n$. Then $\mathcal{A}_\pi$ is said to be *very strongly uniformly elliptic* if it satisfies the uniform Legendre condition, that is, if

$$a_{jk}^{rs}(x)\zeta_r^j\zeta_s^k > 0, \quad x \in \overline{\Omega}, \quad \zeta \in \mathbb{R}^{nN}\backslash\{0\}\ . \tag{10}$$

It is called *uniformly strongly elliptic* if it satisfies the uniform Legendre-Hadamard condition, that is, if

$$a_{jk}^{rs}(x)\xi^j\xi^k\lambda_r\lambda_s > 0, \quad x \in \overline{\Omega}, \quad \xi \in \mathbb{R}^n\backslash\{0\}, \quad \lambda \in \mathbb{R}^N\backslash\{0\}\ .$$

In the following $\mathcal{A}_\pi$ is said to be *normally elliptic* if

$$\sigma(a_{jk}(x)\xi^j\xi^k) \subset [\operatorname{Re} z > 0] := \{z \in \mathbb{C}\,;\ \operatorname{Re} z > 0\}, \quad x \in \overline{\Omega}, \quad \xi \in \mathbb{R}^n\backslash\{0\},$$

where $\sigma(\cdot)$ denotes the spectrum. It is easily seen that the very strong uniform ellipticity implies the strong uniform ellipticity, and that the latter implies the normal ellipticity. If $\mathcal{A}_\pi$ is of separated divergence form, $a_{jk} = \mathbf{A}\alpha_{jk}$, then (10) takes the form

$$\mathbf{A}^{rs}(x)\alpha_{jk}(x)\zeta_r^j\zeta_s^k > 0\ ,$$

which is only easy to check if $\alpha_{jk} = \delta_{jk}$. In the latter case it is equivalent to the requirement that the symmetric part of $\mathbf{A}$ be uniformly positive definite, that is, to

$$\mathbf{A}(x) + \mathbf{A}^T(x) > 0, \quad x \in \overline{\Omega}.$$

The latter condition is always equivalent to the uniform Legendre-Hadamard condition without any restriction upon α. Finally, $\mathcal{A}_\pi$ is normally elliptic if and only if

$$\sigma(\mathbf{A}(x)) \subset [\operatorname{Re} z > 0], \quad x \in \overline{\Omega} ,$$

that is, if and only if all eigenvalues of $\mathbf{A}(x), x \in \overline{\Omega}$, have positive real parts.

The condition of normal ellipticity is in a certain sense optimal. In fact, if we denote by A_p the L_p-realization of the boundary value problem $(\mathcal{A}_\pi, \mathcal{B}_\pi)$, it follows from Theorem 2.4 below that the normal ellipticity of $\mathcal{A}_\pi$ is necessary for $-A_p$ to generate an analytic semigroup on $L_p := L_p(\Omega, \mathbb{R}^N)$, $1 < p < \infty$. Moreover, if $\mathcal{A}_\pi$ is of separated divergence form and δ equals either 0 or 1 on each component of $\partial\Omega$ (for example) then the normal ellipticity of $\mathcal{A}_\pi$ is also sufficient. (A detailed discussion of 'normally elliptic boundary value problems' is given in Section 4 below.) It is well known that the fact that $-A_p$ generates an analytic semigroup is intimately related to regularity properties of solutions to the linear system

$$\begin{aligned} \partial_t u + \mathcal{A}_\pi u &= f \quad \text{in } \Omega \times (0,\infty), \\ \mathcal{B}_\pi u &= 0 \quad \text{on } \partial\Omega \times (0,\infty). \end{aligned}$$

Thus if we loose normal ellipticity we obtain some sort of 'degenerate problem'.

The concept of normal ellipticity is also important from other points of view of applications as is seen by looking at the concrete example (7)-(8). If we put

$$G_{se} := \{\xi \in \mathbb{R}^2\,;\ \mathbf{A}(x,\xi) + \mathbf{A}^T(x,\xi) > 0, x \in \overline{\Omega}\}$$

and

$$G_{ne} := \{\xi \in \mathbb{R}^2\,;\ \sigma(\mathbf{A}(x,\xi)) \subset [\operatorname{Re} z > 0], x \in \overline{\Omega}\}$$

then it is easily verified that $G_{se} \not\supset (\mathbb{R}^+)^2$ if either $\beta_{11} = 0$ and $\beta_{12} \neq 0$, or $\beta_{22} = 0$ and $\beta_{21} \neq 0$, whereas $G_{ne} \supset (\mathbb{R}^+)^2$ without any restriction. Thus, if we were forced to impose the uniform strong ellipticity condition in our example (7) (8), it would not be possible to study solutions with arbitrary nonnegative initial values ($v_0 \geq 0$, $w_0 \geq 0$), in general.

In order to describe some of our main results now we assume that

(i) $(\mathcal{A}(u), \mathcal{B}(u))$ is of separated divergence form, i.e., $a_{jk}(\cdot, u) = \mathbf{A}(\cdot, u)\alpha_{jk}$;
(ii) $\sigma(\mathbf{A}(x,\xi)) \subset [\operatorname{Re} z > 0]$, $(x,\xi) \in \overline{\Omega} \times G$;
(iii) $\delta|\Gamma \in \{0,1\}$ for each component Γ of $\partial\Omega$.

We fix any $p \in (n, \infty)$ and put

$$H^1_{p,\mathcal{B}} := \{u \in H^1_p(\Omega, \mathbb{R}^N)\,;\ (1-\delta)u|\partial\Omega = 0\} .$$

Moreover we define an open subset V of $H^1_{p,\mathcal{B}}$ by putting

$$V := \{v \in H^1_{p,\mathcal{B}}\,;\ v(\overline{\Omega}) \subset G\} .$$

Then we have the following

Theorem. *Given any $u_0 \in V$, there exists a unique maximal solution*

$$u(\cdot, u_0) \in C([0, t^+(u_0)), V) \cap C^\infty(\overline{\Omega} \times (0, t^+(u_0)), \mathbb{R}^N)$$

of

$$\begin{aligned} \partial_t u + \mathcal{A}(u)u &= f(\cdot, u, \partial u) && \text{in } \Omega \times (0,\infty)\ , \\ \mathcal{B}(u)u &= 0 && \text{on } \partial\Omega \times (0,\infty)\ , \\ u(\cdot, 0) &= u_0 && \text{on } \Omega\ , \end{aligned}$$

where $0 < t^+(u_0) \le \infty$. The map

$$(t, u_0) \mapsto u(t, u_0) \tag{11}$$

defines a smooth local semiflow on V such that bounded orbits, which are bounded away from ∂V, are relatively compact in V and bounded in H_p^2 for $t > 0$.

Proof: This follows from Theorem 4.4, Propositions 7.1 and 7.2, Corollaries 7.4 and 9.4, Theorem 10.5, and Remark 10.6 below. ∎

It should be noted that the smoothness of the semiflow refers to the topology of V, of course, thus to the H_p^1-topology. Moreover the Theorem implies that a given solution $u(\cdot, u_0)$ either stays bounded away from the boundary of V, in which case it exists for all time (i.e., $t^+(u_0) = \infty$) and has a nonempty ω-limit set, or converges to ∂V (in finite or infinite time). If G is maximal in the sense that $\xi \in G$ if and only if $-\partial_j(a_{jk}(\cdot, \xi)\partial_k \cdot)$ is normally elliptic, then the convergence of $u(\cdot, u_0)$ to ∂V in finite time implies that the problem 'degenerates' at $t = t^+(u_0)$.

It should also be noted that the Theorem contains the strong assertion that a solution, which is bounded in H_p^1, is already bounded in H_p^2 for $t \in [\varepsilon, \infty)$, $\varepsilon > 0$, provided it does not reach ∂V. Moreover, it is a consequence of Corollary 9.4 below that each weak solution in the H_p^1-sense is already a smooth solution (in the $C^\infty(\overline{\Omega} \times (0, t^+), \mathbb{R}^N)$ sense). In other words, if 'blow-up' does occur, it has to occur in the H_p^1-topology (and not in a higher norm). In fact, in the forthcoming third paper of the present series it will be shown that 'blow-up' has to occur in a much weaker norm. In other words, it will be shown that the boundedness of $u(\cdot, u_0)$ in much weaker norms (in L_∞-norms in some cases) implies already global existence of smooth solutions.

The results derived below are much more general in many respects:

- The regularity assumptions can be considerably weakened, and the coefficients of $(\mathcal{A}(u), \mathcal{B}(u))$, as well as the nonlinearity f, can be nonlocal operators.
- The growth restrictions with respect to the gradient can be weakened to admit arbitrary polynomial growth (at the expense that one has to replace H_p^1 by H_p^s for some $s \in (1, 1 + 1/p)$).
- $(\mathcal{A}(u), \mathcal{B}(u))$ can be a general normally elliptic system as introduced in Section 1 and discussed in detail in Section 4 below.
- The smooth dependence of the solution on additional parameters, which may vary in an open subset of an arbitrary Banach space, is investigated.

- It is shown that the linearization of the semiflow is the unique solution of the (naturally) linearized system.
- There are more general conditions than the one contained (implicitly) in the Theorem, guaranteeing that $u(\cdot, u_0)$ is a global solution.
- The assumption that $g(\cdot, 0) = 0$ can be dropped.
- Corresponding results are true for the nonautonomous case.

If we apply the above Theorem to the concrete Example (7) (8) with $G := G_{ne}$, we see that there exists for each $u_0 := (v_0, w_0) \in V$ a unique smooth solution $u := u(\cdot, u_0)$. Observe that we can rewrite (7) (8) in the form

$$\begin{aligned} \partial_t u^r - a^r(\cdot, u)\Delta u^r + d_j^r(\cdot, u, \partial u)\partial_j u^r &= u^r e^r(\cdot, u, \partial u, \Delta u) \quad &&\text{in } \Omega \times (0, \infty) , \\ a^r(\cdot, u)\partial_\nu u^r + b^r(\cdot, u, \partial_\nu u)u^r &= 0 &&\text{on } \partial\Omega \times (0, \infty) \end{aligned}$$

for $r = 1, 2$ (no summation over r), where a^r, b^r, d_j^r, and e^r are smooth functions of their arguments. This shows — by inserting the already found solution u in a^r, b^r, d_j^r, e^r — that each component $u^r(\cdot, u_0)$ of $u(\cdot, u_0)$ is a smooth solution of a linear parabolic initial boundary value problem of the form

$$\begin{aligned} \partial_t \omega - a(\cdot, t)\Delta\omega + d_j(\cdot, t)\partial_j \omega &= e(\cdot, t)\omega \quad &&\text{in } \Omega \times (0, t^+) , \\ a(\cdot, t)\partial_\nu \omega + b(\cdot, t)\omega &= 0 &&\text{on } \partial\Omega \times (0, t^+) , \\ \omega(\cdot, 0) &= \omega_0 &&\text{on } \Omega , \end{aligned} \tag{12}$$

where $t^+ := t^+(u_0)$ and $a(x, t) \geq \alpha > 0$ on $\overline{\Omega} \times [0, t^+)$. Now we can apply the maximum principle to (12) to deduce that $u(t, u_0) \in P := \{v \in H_p^1 \, ; \, v(\overline{\Omega}) \subset (\mathbb{R}^+)^2\}$ if $u_0 \in P$. (The fact that $b(\cdot, t) \ngeq 0$, in general, requires a modification of the maximum principle along the lines of [5, Section 6] by means of Theorem B.3 of the Appendix below.) This shows that P is positively invariant for the semiflow generated by (7) (8). Hence it follows that

> the reaction-diffusion system (7) (8) possesses for each nonnegative initial value $u_0 \in P$ a unique maximal smooth solution $u(\cdot, u_0)$, and $u(\cdot, u_0)$ is always nonnegative, that is, $u(t, u_0) \in P$ for $0 \leq t < t^+(u_0)$.

It is not difficult to see that P is bounded away from ∂V. Hence the Theorem implies that $u(\cdot, u_0)$ is a global solution of (7) (8) for $u_0 \in P$, provided we can show that

$$\sup_{0 < t < t^+(u_0)} \|u(t, u_0)\|_{1,p} < \infty .$$

In this case $t^+(u_0) = \infty$,

$$\sup_{\varepsilon < t < \infty} \|u(t, u_0)\|_{2,p} < \infty, \quad \varepsilon > 0 ,$$

and the orbit $\{u(t, u_0) \, ; \, 0 \leq t < \infty\}$ is relatively compact in H_p^1, so that it has a nonempty ω-limit set.

So far the existence of positive solutions for the ecology problem (7) (8) is available in the literature under very restrictive additional hypotheses only. In all results known to the author $\alpha_1, \alpha_2, \beta_{11}, \dots, \beta_{22}$ are constants and $\gamma_1 = \gamma_2 = 0$. More precisely, Kim [25] proved the local existence of smooth positive solutions if $n = 1$ and $\beta_{11} = \beta_{22} = 0$, and if h_r is independent of $x \in \overline{\Omega}$ and affine in $\xi \in \mathbb{R}^2$ (with negative partial derivatives). If, in addition, $\alpha_1 = \alpha_2$, he obtained global existence. Deuring [17] established global existence of classical positive solutions under the same hypotheses as Kim, except that he did not require $n = 1$ or $\alpha_1 = \alpha_2$. However he had to impose size restrictions upon the various coefficients, depending on the size of the initial values. Finally, Pozio and Tesei [34] prove an existence theorem for (7) under Dirichlet boundary conditions, assuming $\alpha_1 = \alpha_2 = 1$ and $\beta_{21} = 0$ (which reduces $\mathbf{A}$ to a triangular matrix). They allow nonlinear functions h_r possessing a certain prescribed qualitative behaviour. Moreover, given an additional superlinearity condition for h_1, they obtain global solutions.

This paper generalizes and extends considerably the results for general second order quasilinear parabolic systems in divergence form contained in [11]. In the latter paper we had not been able to prove that the solutions to problem (6) generate a semiflow on V, whereas we obtain now even smoothness results of this semiflow for much more general systems, as well as a lot of additional information.

There are other approaches to quasilinear parabolic equations and systems, which are based on different techniques. In particular they use Sobolev spaces of higher order and Hölder space theory and apply also to equations which are not in divergence form [1] (or are even 'fully nonlinear' as in [28] in the case of a single equation). The advantage of our theory lies in the fact that we can work in a rather weak and natural setting, namely in an open subset of the Banach space $H^1_{p,\mathcal{B}}$ for any $p > n$. This is of great importance for a qualitative study of the semiflow since it is much easier to study a semiflow on an open subset of a Banach space than on a Banach manifold. (Such a nonlinear structure comes in automatically through compatibility conditions involving nonlinear boundary conditions, which are necessary if one works in spaces with more regularity than H^1_p, say in H^2_p- or $C^{2+\alpha}$-spaces.) Moreover, our Banach spaces are reflexive and much better suited than Hölder spaces for general techniques of nonlinear functional analysis, which may be useful for further investigations of the structure of the semiflow generated by these problems. In addition, it is sometimes possible to find a priori bounds in some 'weak norm' (say L_q-bounds) which may be useful to establish the global existence of a given solution. For this one has to be able to work in 'weak spaces' and to take advantage of the smoothing property of the 'parabolic' semiflow generated by (11). This question will be attacked in the forthcoming paper mentioned already above. For those investigations the present H^1_p-setting is a most important prerequisite.

In this connection it should be mentioned that it is, in general, impossible to work in a (superficially more natural) H^1_2-setting, due to the well known fact that weak solutions (in the H^1_2-sense) of parabolic systems are not Hölder continuous, in general (e.g. [22]). To avoid these difficulties we have chosen the (technically more complicated) H^1_p-setting for $p > n$, which builds in the Hölder regularity from the very beginning.

Finally it should be mentioned that the 'classical methods', which are based upon the 'test function technique' and which have been used in [23], do not seem to apply to the case of normally elliptic systems. In fact, they seem to be restricted to operators which are very strongly uniformly elliptic.

This paper consists of three parts and an Appendix. In Part One we give a thorough discussion of second order normally elliptic systems. In particular we topologize the set of all systems of this type under minimal regularity assumptions. In Part Two we introduce the extrapolation setting, which is the main technical device in our approach. Part Three contains the main results of this paper, namely the general existence, continuity and smoothness assertions indicated above.

In Appendix A we give an extension of the main results of [13] to the case of abstract parabolic equations on all of $\mathbb{R}^+$. These results are needed to prove the boundedness of solutions in H_p^2, given their boundedness in H_p^1. In Appendix B we prove a general 'extension theorem of boundary data' which is needed to deal with inhomogeneous boundary conditions, and which is of independent interest.

The present paper depends, of course, on the first paper of this series [15], which contains the 'soft part', so to speak. It depends also crucially upon [13], since the latter paper provides the tools for the proof of the H_p^2-regularity of the H_p^1-solutions, which are found by the extrapolation techniques. As mentioned already above, the more 'classical' test function techniques, which are usually employed to prove (partial) regularity results (e.g. [22]) do not apply to our situation in which we consider general normally elliptic systems.

Finally it should be clear that the results of this paper can be extended to higher order systems which possess an appropriate divergence form structure. For simplicity — and for their importance in applications — we have restricted our considerations to second order systems.

Part One: Linear reaction-diffusion systems.

1. Second order normally elliptic boundary value problems. Throughout Part one of this paper we denote by Ω a bounded domain in $\mathbb{R}^n$ of class C^2, that is, $\overline{\Omega}$ is a compact n-dimensional C^2-submanifold of $\mathbb{R}^n$ with boundary $\partial\Omega$. We write Γ for the set of components Γ of $\partial\Omega$, and $T(\partial\Omega)$ denotes the tangent bundle of $\partial\Omega$.

If E and F are Banach spaces (over $\mathbb{K} := \mathbb{R}$ or $\mathbb{C}$), we denote by $\mathcal{L}(E,F)$ the Banach space of bounded linear operators from E to F, and $\mathrm{Isom}\,(E,F)$ is the set of isomorphisms in $\mathcal{L}(E,F)$. Moroever, $\mathcal{L}(E) := \mathcal{L}(E,E)$ and $\mathcal{GL}(E) := \mathrm{Isom}\,(E,E)$. The identity in the Banach algebra $\mathcal{L}(E)$ is usually denoted by 1, and $\mathcal{L}(\mathbb{K}^N)$ is always identified with $\mathbb{K}^{N\times N}$, the space of $(N \times N)$-matrices, by means of the canonical basis of $\mathbb{K}^N$.

If A is a linear operator in a Banach space, with domain $D(A)$, then $\sigma(A)$ denotes the spectrum and $\rho(A)$ the resolvent set (of its complexification if $\mathbb{K} = \mathbb{R}$). More generally, if $\mathbb{K} = \mathbb{R}$ and complex quantities occur in a given formula, it is always understood that we refer to the corresponding complexifications.

We denote by $\mathcal{A} := \mathcal{A}(x,\partial)$ a linear second order differential operator acting on N-vector valued functions $u : \Omega \to \mathbb{K}^N$, that is,

$$\mathcal{A} := -a_{jk}\partial_j\partial_k + a_j\partial_j + a_0, \tag{1.1}$$

where $\partial_j := \partial/\partial x^j$ and

$$a_{jk} \in C(\overline{\Omega}, \mathcal{L}(\mathbb{K}^N)), \quad a_j, a_0 \in L_1(\Omega, \mathcal{L}(\mathbb{K}^N)), \ \ 1 \le j,k \le n\ . \tag{1.2}$$

We write a_π for the principal symbol of $\mathcal{A}$, that is,

$$a_\pi(x,\xi) := a_{jk}(x)\xi^j\xi^k\ ,$$

so that

$$a_\pi \in C(\overline{\Omega} \times \mathbb{R}^n, \mathcal{L}(\mathbb{K}^N)) . \tag{1.3}$$

Then $\mathcal{A}$ is said to be *normally elliptic* if

$$\sigma(a_\pi(x,\xi)) \subset [\operatorname{Re} z > 0], \quad (x,\xi) \in \overline{\Omega} \times (\mathbb{R}^n \backslash \{0\}) . \tag{1.4}$$

We fix a function

$$(\delta^1, \dots, \delta^N) \in C(\partial\Omega, \{0,1\})^N$$

and denote the (constant) value of δ^r on Γ by $\delta^r(\Gamma)$. Moreover we put

$$\delta := \operatorname{diag}[\delta^r]_{1 \le r \le N} \in C(\partial\Omega, \mathcal{L}(\mathbb{K}^N)) . \tag{1.5}$$

We assume that

$$b_0, b_j, c \in C(\partial\Omega, \mathcal{L}(\mathbb{K}^N)), \quad 1 \le j \le n , \tag{1.6}$$

and define a "boundary operator" $\mathcal{B} := \mathcal{B}(x, \partial)$ by

$$\mathcal{B} := \delta(b_j \gamma \partial_j + b_0 \gamma) + (1 - \delta) c \gamma , \tag{1.7}$$

where γ denotes the trace operator: $\gamma u := u | \partial\Omega$. It should be observed that every system of N linear differential operators of order at most 1 on $\partial\Omega$ can be written in the form (1.7).

We associate with $\mathcal{B}$ the 'principal boundary symbol'

$$b_\pi \in C(\partial\Omega \times \mathbb{R}^n, \mathcal{L}(\mathbb{K}^N)) , \tag{1.8}$$

defined by

$$b_\pi(\cdot, \xi) := \delta b_j \xi^j + (1 - \delta) c, \quad \xi \in \mathbb{R}^n . \tag{1.9}$$

Then $\mathcal{B}$ is said to satisfy the *normal complementing condition with respect to* $\mathcal{A}$ (of Lopatinskii-Shapiro type) if zero is, for each $(x, \xi) \in T(\partial\Omega)$ and $\lambda \in [\operatorname{Re} z \ge 0]$ with $(\xi, \lambda) \ne (0,0)$, the only exponentially decaying solution of the boundary value problem on the half line:

$$[\lambda + a_\pi(x, \xi + \nu(x) i \partial_t)] u = 0, \quad b_\pi(x, \xi + \nu(x) i \partial_t) u(0) = 0, \quad t > 0. \tag{1.10}$$

Finally, $(\mathcal{A}, \mathcal{B})$ is [a] *normally elliptic* [*boundary value problem*] (on Ω) if $\mathcal{A}$ is normally elliptic and $\mathcal{B}$ satisfies the normal complementing condition with respect to $\mathcal{A}$.

The importance of the concept of normally elliptic boundary value problems follows from Theorem 2.4 below. In Section 4 we shall give a number of explicit conditions guaranteeing that $(\mathcal{A}, \mathcal{B})$ is normally elliptic.

2. A priori estimates. It is the purpose of the following considerations to topologize the set of second order normally elliptic boundary value problems. It is clear that the latter set is decomposed in finitely many disjoint subsets, which are distinguished by the function δ of (1.5). Hence it suffices to consider each one of these subclasses separately. Consequently we fix δ throughout the rest of this paper.

We put

$$E(\Omega) := C(\overline{\Omega}, \mathcal{L}(\mathbb{K}^N))^{n^2} \times C(\partial\Omega, \mathcal{L}(\mathbb{K}^N))^n \times C(\partial\Omega, \mathcal{L}(\mathbb{K}^N))$$

and denote the general point of $E(\Omega)$ by

$$e := ((a_{jk}), (b_1, \dots, b_n), c) ,$$

where we order the n^2-tuple (a_{jk}) lexicographically. Given $e \in E(\Omega)$, we put

$$\mathcal{A}_\pi(e) := -a_{jk}\partial_j\partial_k, \quad \mathcal{B}_\pi(e) := \delta b_j \gamma \partial_j + (1-\delta)c ,$$

and

$$\mathcal{E}(\Omega) := \{e \in E(\Omega)\,;\ (\mathcal{A}_\pi(e), \mathcal{B}_\pi(e)) \text{ is normally elliptic}\} .$$

Then we have the basic

Theorem 2.1. *$\mathcal{E}(\Omega)$ is open in $E(\Omega)$.*

Proof: Given $e \in E(\Omega)$, we denote by $a_\pi(e)$ and $b_\pi(e)$ the (principal) symbol of $\mathcal{A}_\pi(e)$ and $\mathcal{B}_\pi(e)$, respectively. Moreover we write $\eta := \sqrt{\lambda}$ for $\lambda \in [\operatorname{Re} z \geq 0]$, using the principal value of the square root. Then $e \in \mathcal{E}(\Omega)$ if and only if zero is, for each $(x, \xi) \in T(\partial\Omega)$ and $\eta \in S := [|\arg z| \leq \pi/4] \cup \{0\}$ with $(\xi, \eta) \neq (0,0)$, the only exponentially decaying solution of

$$[\eta^2 + a_\pi(e)(x, \xi + \nu(x)i\partial_t)]u = 0, \quad b_\pi(e)(x, \xi + \nu(x)i\partial_t)u(0) = 0, \quad t > 0. \tag{2.1}$$

Observe that

$$(\xi, \eta) \mapsto \eta^2 + a_\pi(e)(x, \xi)$$

and

$$(\xi, \eta) \mapsto \delta b_\pi(e)(x, \xi)$$

are positively homogeneous on $T_x(\partial\Omega) \times S$ of degree 2 and 1, respectively. Since

$$\alpha\xi + \nu(x)i\partial_t = \alpha[\xi + \nu(x)i\partial_{\alpha t}], \quad \alpha > 0 ,$$

it follows that it suffices to consider (2.1) on

$$\Sigma := \{(x, \xi, \eta) \in \partial\Omega \times \mathbb{R}^n \times \mathbb{C}\,;\ (\xi, \eta) \in \Sigma_x\} ,$$

where

$$\Sigma_x := \{(\xi, \eta) \in T_x(\partial\Omega) \times S\,;\ |\xi|^2 + |\eta|^2 = 1\} .$$

Observe that Σ is compact.

Put $\tilde{a}(e)(\sigma) := \eta^2 + a_\pi(e)(x, \xi)$ for $\sigma \in \Sigma$ and suppose that $e_0 \in \mathcal{E}(\Omega)$. Then (1.4) implies

$$[\sigma \mapsto \tilde{a}_\pi(e_0)(\sigma)] \in C(\Sigma, \mathcal{GL}(\mathbb{C}^N)) .$$

Since $\mathcal{GL}(\mathbb{C}^N)$ is open in $\mathcal{L}(\mathbb{C}^N)$ and Σ is compact, it follows that $C(\Sigma, \mathcal{GL}(\mathbb{C}^N))$ is open in $C(\Sigma, \mathcal{L}(\mathbb{C}^N))$. Since

$$[e \mapsto \tilde{a}_\pi(e)] \in C(E(\Omega), C(\Sigma, \mathcal{L}(\mathbb{C}^N))) ,$$

we can find a neighbourhood U of e_0 in $E(\Omega)$ so that

$$\tilde{a}_\pi(U) \subset C(\Sigma, \mathcal{GL}(\mathbb{C}^N)) .$$

For each $e \in U$ we can rewrite the system (2.1) for a given $\sigma \in \Sigma$ in the equivalent form

$$\begin{aligned} D^2u - A_1(e,\sigma)Du - A_0(e,\sigma)u = 0, \quad t > 0, \\ B_1(e,\sigma)Du(0) + B_0(e,\sigma)u(0) = 0, \end{aligned} \tag{2.2}$$

where $D := -i\partial_t$,

$$[e \mapsto A_j(e,\cdot)] \in C(U, C(\Sigma, \mathcal{L}(\mathbb{C}^N))), \quad \text{and} \quad [e \mapsto B_j(e,\cdot)] \in C(U, C(\Sigma, \mathcal{L}(\mathbb{C}^N)))$$

for $j = 0,1$. By the standard reduction of a second order ordinary differential equation to a first order system we see that the first equation in (2.2) is equivalent to the first order equation

$$\dot{v} = i\mathbb{A}(e,\sigma)v \quad \text{in} \quad Y := \mathbb{C}^N \times \mathbb{C}^N \tag{2.3}$$

by means of the identification $v := (u, Du)$, where $\mathbb{A}(e,\sigma)$ has the block matrix representation in $\mathcal{L}(\mathbb{C}^N \times \mathbb{C}^N)$

$$\begin{bmatrix} 0 & 1 \\ A_0(e,\sigma) & A_1(e,\sigma) \end{bmatrix} .$$

Thus

$$[e \mapsto \mathbb{A}(e,\cdot)] \in C(U, C(\Sigma, \mathcal{L}(Y))) . \tag{2.4}$$

By expanding the above matrix with respect to the last N rows it can be shown that

$$\det(z - \mathbb{A}) = \det(z^2 - A_1 z - A_0)$$

for $z \in \mathbb{C}$ (cf. [24, Lemma 2.1] where a much more general case is treated). Since the right hand side equals

$$[\det\, a_\pi(e,\sigma)]^{-1} \det(\lambda + a_\pi(x, \xi - \nu(x)z)) ,$$

we see that

$$z \in \sigma(\mathbb{A}(e,\sigma)) \Longleftrightarrow -\lambda \in \sigma(a_\pi(x, \xi - \nu(x)z)) .$$

Thus we deduce from (1.4) that

$$\mathbb{R} \subset \rho(\mathbb{A}(e,\sigma)), \quad (e,\sigma) \in U \times \Sigma . \tag{2.5}$$

Let $x \in \partial\Omega$ and $e \in U$ be fixed and denote by $m_\pm(\xi,\eta)$ the number of roots of the polynomial

$$z \mapsto \det(\lambda + a_\pi(x, \xi - \nu(x)z)) \tag{2.6}$$

in $[\pm \operatorname{Im} z > 0]$. Observe that (2.5) implies $m_+(\xi,\eta) + m_-(\xi,\eta) = 2N$. Since Σ_x is connected, we deduce by Rouché's theorem that $m_\pm(\xi,\eta) = m_\pm(0,1)$. Since z is a root of (2.6) for $(\xi,\eta) = (0,1)$ if and only if

$$-1 \in \sigma(a_\pi(x, \nu(x)z)) = \sigma(a_\pi(x, \nu(x)(-z))) ,$$

we see that $m_+(0,1) = m_-(0,1)$. Hence $m_+(\xi,\eta) = N$.

It follows from (2.5) that (2.3) generates a hyperbolic linear flow on Y:

$$e^{i\mathbb{A}(e,\sigma)t}y, \quad t \in \mathbb{R}, \quad y \in Y . \tag{2.7}$$

Hence Y decomposes into the direct sum $Y = Y_s(e,\sigma) \oplus Y_u(e,\sigma)$, where $Y_s(e,\sigma)$ is the stable and $Y_u(e,\sigma)$ the unstable subspace of (2.7). Moreover the above considerations show that $\dim Y_s(e,\sigma) = N$ (cf. [6, Section 13]).

Let γ denote a positively oriented closed smooth Jordan curve in $[\operatorname{Im} z > 0]$, containing $\sigma(\mathbb{A}(e,\sigma)) \cap [\operatorname{Im} z > 0]$ in its interior. Then

$$P(e,\sigma) := \frac{1}{2\pi i}\int_\gamma (z - \mathbb{A}(e,\sigma))^{-1}\,dz \tag{2.8}$$

is the projection onto $Y_s(e,\sigma)$, parallel to $Y_u(e,\sigma)$. Since Σ is compact and the spectrum is upper semicontinuous, it follows from (2.4) that we can assume (by making U smaller, if necessary) that (2.8) is true for all $(e,\sigma) \in U \times \Sigma$, where γ is fixed. Then we deduce from (2.4), the continuity of the inversion map, and (2.8) that

$$[e \mapsto P(e,\cdot)] \in C(U, C(\Sigma, \mathcal{L}(Y))) .$$

Define

$$[e \mapsto \mathbb{B}(e,\cdot)] \in C(U, C(\Sigma, \mathcal{L}(Y, \mathbb{C}^N))) \tag{2.9}$$

by

$$\mathbb{B}(e,\sigma)y := B_0(e,\sigma)y_1 + B_1(e,\sigma)y_2, \quad y := (y_1,y_2) \in Y = \mathbb{C}^N \times \mathbb{C}^N .$$

Then it follows from the above considerations that u is an exponentially decaying solution of (2.1) if and only if $u = pr_1 v$, where $pr_1 : \mathbb{C}^N \times \mathbb{C}^N \to \mathbb{C}^N$ is the natural projection onto the first factor, and

$$v(t) = e^{i\mathbb{A}(e,\sigma)t}y, \quad y \in Y_s(e,\sigma), \quad \mathbb{B}(e,\sigma)y = 0 .$$

Thus zero is the only exponentially decaying solution of (2.1) iff $\mathbb{B}(e,\sigma)|Y_s(e,\sigma)$ is injective. Since $\dim Y_s(e,\sigma) = N$, the latter is the case iff $\mathbb{B}(e,\sigma)|Y_s(e,\sigma)$ is an isomorphism from $Y_s(e,\sigma)$ onto $\mathbb{C}^N$, which is, in turn, equivalent to

$$\mathbb{B}P(e,\sigma) := \mathbb{B}(e,\sigma)P(e,\sigma) \in \mathcal{SL}(Y, \mathbb{C}^N) ,$$

where $\mathcal{SL}(Y,\mathbb{C}^N)$ is the set of all surjections in $\mathcal{L}(Y,\mathbb{C}^N)$.

Observe that

$$[e \mapsto \mathbb{B}P(e,\cdot)] \in C(U, C(\Sigma, \mathcal{L}(Y, \mathbb{C}^N))) \tag{2.10}$$

by (2.8) and (2.9), and that

$$\mathbb{B}P(e_0, \Sigma) \subset \mathcal{SL}(Y, \mathbb{C}^N)$$

since $e_0 \in \mathcal{E}(\Omega)$. It follows from (2.10) that $\mathbb{B}P(e_0,\Sigma)$ is a compact subset of the open subset $\mathcal{SL}(Y,\mathbb{C}^N)$ of $\mathcal{L}(Y,\mathbb{C}^N)$. Hence we deduce from (2.10) that we can assume — by making U again smaller, if necessary — that $\mathbb{B}P(U\times\Sigma)\subset\mathcal{SL}(Y,\mathbb{C}^N)$, which means that $U\subset\mathcal{E}(\Omega)$. This proves the theorem. ∎

Given $p\in(1,\infty)$, we fix $\hat{p}\in\mathbb{R}$ so that

$$\hat{p}=\begin{cases} p\vee p' & \text{if } p\vee p'>n,\\ >n & \text{otherwise}\end{cases}$$

where $p':=p/(p-1)$. We write H_p^s and $B_{p,p}^s$, $s\in\mathbb{R}$, for the standard Bessel potential and Besov spaces, respectively (e.g. [43]). Recall that $H_p^0=L_p$ and that H_p^s coincides, except for equivalent norms, with the usual Sobolev spaces W_p^s, provided $s\in\mathbb{N}$.

We put

$$\begin{aligned}\mathbb{E}_p(\Omega):=C(\overline{\Omega},X)^{n^2}\times L_{\hat{p}}(\Omega,X)^n\times L_{\hat{p}}(\Omega,X)\times B_{\hat{p},\hat{p}}^{1-1/\hat{p}}(\partial\Omega,X)^n\\ \times B_{\hat{p},\hat{p}}^{1-1/\hat{p}}(\partial\Omega,X)\times B_{\hat{p},\hat{p}}^{2-1/\hat{p}}(\partial\Omega,X),\end{aligned}$$

where $X:=\mathcal{L}(\mathbb{K}^N)$ and denote the general point of $\mathbb{E}_p(\Omega)$ by

$$\underset{\sim}{e}:=((a_{jk}),(a_1,\dots,a_n),a_0,(b_1,\dots,b_n),b_0,c)\ .$$

Since $\hat{p}>n$, Sobolev's theorem implies the continuity of the linear projection

$$\pi:\mathbb{E}_p(\Omega)\to E(\Omega),\quad \underset{\sim}{e}\mapsto e:=((a_{jk}),(b_1,\dots,b_n),c)\ .$$

Hence it follows from Theorem 2.1 that

$$\mathcal{E}_p(\Omega):=\pi^{-1}\mathcal{E}(\Omega)\ \text{ is open in }\ \mathbb{E}_p(\Omega)\ .$$

Throughout the remainder of this paper

$$H_p^s:=(H_p^s(\Omega,\mathbb{K}^N),\|\cdot\|_{s,p}),\quad s\in\mathbb{R},\ 1<p<\infty\ ,$$

and $\|\cdot\|_p:=\|\cdot\|_{o,p}$. Moreover,

$$\partial B_p^s:=\prod_{\Gamma\in\mathfrak{\Gamma}}\prod_{r=1}^{N}B_{p,p}^{s-\delta^r(\Gamma)-1/p}(\Gamma,\mathbb{K})\ .$$

We define a linear map

$$(\mathcal{A}_p(\cdot),\mathcal{B}_p(\cdot)):\mathbb{E}_p(\Omega)\to\mathcal{L}(H_p^2,L_p\times\partial B_p^2)\cap\mathcal{L}(H_{p'}^2,L_{p'}\times\partial B_{p'}^2)$$

by putting

$$\mathcal{A}_p(\underset{\sim}{e}):=-a_{jk}\partial_j\partial_k+a_j\partial_j+a_0$$

and

$$\mathcal{B}_p(\underset{\sim}{e}):=\delta(b_j\gamma\partial_j+b_0\gamma)+(1-\delta)c\gamma\ .$$

Observe that $(\mathcal{A}_p(\underset{\sim}{e}),\mathcal{B}_p(\underset{\sim}{e}))$ is normally elliptic if and only if $\underset{\sim}{e}\in\mathcal{E}_p(\Omega)$.

Lemma 2.2. $(\mathcal{A}_p(\cdot), \mathcal{B}_p(\cdot))$ *is continuous.*

Proof: This is an easy consequence of Sobolev's imbedding theorem and Hölder's inequality. ▮

Given $\omega > 0$ and $\vartheta \in [0, \pi/2]$, we put

$$S(\vartheta, \omega) := [|\arg z| \leq \vartheta + \pi/2] \cap [|z| \geq \omega] .$$

Moreover

$$\beta(\lambda, q) := \mathrm{diag}[1 + |\lambda|^{(2-\delta^r-1/q)/2}]_{1 \leq r \leq N} \in \mathcal{L}(\mathbb{C}^N)$$

for $\lambda \in \mathbb{C}$ and $1 < q < \infty$. Using these notations we can formulate the following fundamental

Theorem 2.3. *Let B be a bounded subset of $\mathcal{E}_p(\Omega)$ so that $\pi(B)$ is relatively compact in $\mathcal{E}(\Omega)$. Then there exist a neighbourhood U of B in $\mathbb{E}_p(\Omega)$, a number $\vartheta \in (0, \pi/2)$, and positive constants κ and ω so that*

(i) $(\lambda + \mathcal{A}_q(\underset{\sim}{e}), \mathcal{B}_q(\underset{\sim}{e})) \in \mathit{Isom}\,(H_q^2, L_q \times \partial B_q^2)$,

(ii) $\kappa^{-1} \leq \frac{\|(\lambda + \mathcal{A}_q(\underset{\sim}{e}))u\|_q + \|\beta(\lambda,q)\mathcal{B}_q(\underset{\sim}{e})u\|_{\partial B_q^2}}{|\lambda|\|u\|_q + \|u\|_{2,q}} \leq \kappa$ *for* $e \in U, \lambda \in S(\vartheta, \omega), q \in \{p, p'\}$, *and* $u \in H_q^2 \backslash \{0\}$.

Proof: In principle the assertion could be obtained from the L_q-estimates for general elliptic systems of Agmon-Douglis- Nirenberg [4], by studying carefully the minimal regularity assumptions, which are needed, by additional considerations, which are based on 'Agmon's trick' [3], to yield the λ-dependence, and by further considerations guaranteeing the surjectivity of the map under discussion (cf. the investigations in [7, Section 12] and [8, Section 6] and also [20, 21]). A much more transparent proof — which applies also to other situations — can be based upon the Mikhlin multiplier theorem, semigroup theory, and the use of parameter-dependent norms and will be given elsewhere. ▮

Given $\underset{\sim}{e} \in \mathbb{E}_p(\Omega)$, we put

$$H_{q,\mathcal{B}}^2 := \mathit{ker}\, \mathcal{B} = \{u \in H_q^2 \,;\, \mathcal{B}u = 0\}, \quad q \in \{p, p'\} .$$

Then we define an unbounded linear operator $A_q := A_q(\underset{\sim}{e})$ in L_q by

$$A_q : H_{q,\mathcal{B}}^2 \subset L_q \to L_q, \quad u \mapsto \mathcal{A}_q u ,$$

the $\mathbf{L_q}$-*realization of* $(\mathcal{A}_\mathbf{q}, \mathcal{B}_\mathbf{q})$. Observe that A_q is densely defined since $\mathcal{D} := \mathcal{D}(\Omega, \mathbb{C}^N) \subset D(A_q)$, and $\mathcal{D}$, the space of $\mathbb{C}^N$-valued test functions in Ω, is dense in L_q. It follows from Theorem 2.3 that there exist $\vartheta \in (0, \pi/2)$ and positive constants κ and ω so that

$$\rho(-A_q) \supset S(\vartheta, \omega) \tag{2.11}$$

and

$$\kappa^{-1} \leq \frac{\|(\lambda + A_q)u\|_q}{|\lambda|\|u\|_q + \|u\|_{2,q}} \leq \kappa, \quad \lambda \in S(\vartheta, \omega), \ u \in H_{q,\mathcal{B}}^2 \backslash \{0\}, \ q \in \{p, p'\}, \tag{2.12}$$

provided $(\mathcal{A}_p, \mathcal{B}_p)$ is normally elliptic. Observe that (2.12) implies also that A_q is a closed linear operator.

Let X be a Banach space. Then we write $A \in \mathcal{H}(X)$ if $-A$ is the infinitesimal generator of a strongly continuous analytic semigroup $\{e^{-tA}\,;\, t \geq 0\}$ on X, that is, in $\mathcal{L}(X)$. Recall that $A \in \mathcal{H}(X)$ if and only if A is closed, densely defined, and there exist $\vartheta \in (0, \pi/2)$ and positive constants M and ω so that $S(\vartheta, \omega) \subset \rho(-A)$ and

$$\|(\lambda + A)^{-1}\| \leq M/(1 + |\lambda|), \quad \lambda \in S(\vartheta, \omega)\ . \tag{2.13}$$

Using these facts we can now prove

Theorem 2.4. *If $(\mathcal{A}_p, \mathcal{B}_p)$ is normally elliptic then $A_q \in \mathcal{H}(L_q), q \in \{p, p'\}$. Conversely, if $A_p \in \mathcal{H}(L_p)$ then $\mathcal{A}_p$ is normally elliptic.*

Proof: The first part follows directly from Theorem 2.3 and the above remarks, since (2.12) implies (2.13).

Suppose now that $A := A_p \in \mathcal{H}(L_p)$. Then there exist $\vartheta \in (0, \pi/2)$ and positive constants M and ω so that $S(\vartheta, \omega) \subset \rho(-A)$ and

$$|\lambda| \|u\|_p \leq M \|(\lambda + A)u\|_p, \quad u \in H^2_{p,\mathcal{B}}, \quad \lambda \in S(\vartheta, \omega)\ . \tag{2.14}$$

This implies that $\omega + A \in \operatorname{Isom}(H^2_{p,\mathcal{B}}, L_p)$. Hence, denoting by α positive constants, which may be different from formula to formula, but are always independent of the independent variables occuring at a given place, it follows that

$$\begin{aligned} \|u\|_{2,p} &\leq \alpha \|(\omega + A)u\|_p = \alpha \|(\omega + A)(\lambda + A)^{-1}(\lambda + A)u\|_p \\ &\leq \alpha \|(\lambda + A)u\|_p, \quad \lambda \in S(\vartheta, \omega),\ u \in H^2_{p,\mathcal{B}}, \end{aligned} \tag{2.15}$$

since $(\omega + A)(\lambda + A)^{-1} = (\omega - \lambda)(\lambda + A)^{-1} + 1$ and (2.14) imply

$$\|(\omega + A)(\lambda + A)^{-1}\|_{\mathcal{L}(L_p)} \leq \alpha, \quad \lambda \in S(\vartheta, \omega)\ .$$

Hence we obtain from (2.14) and (2.15) that

$$|\lambda| \|u\|_p + \|u\|_{2,p} \leq \alpha \|(\lambda + A)u\|_p, \quad \lambda \in S(\vartheta, \omega), \quad u \in H^2_{p,\mathcal{B}}\ . \tag{2.16}$$

Given $x_0 \in \overline{\Omega}$, we put $\mathcal{A}' := \mathcal{A} - \mathcal{A}_\pi(x_0, \partial)$. By using the continuity of the top order coefficients of $\mathcal{A}'$ and imbedding and interpolation inequalities for the lower order terms we find, for each $\varepsilon > 0$, a constant $c(\varepsilon)$ and a neighbourhood U_{x_0} of x_0 *in $\overline{\Omega}$ so that*

$$\|\mathcal{A}'u\|_p \leq \varepsilon \|u\|_{2,p} + c(\varepsilon)\|u\|_p, \quad u \in \mathcal{D}(U_{x_0}, \mathbb{C}^N)\ . \tag{2.17}$$

Hence we deduce from (2.16) and (2.17) an estimate of the form

$$|\lambda| \|u\|_p + \|u\|_{2,p} \leq \alpha(\|(\lambda + \mathcal{A}_\pi(x_0, \partial))u\|_p + \|u\|_p) \tag{2.18}$$

for $\lambda \in S(\vartheta, \omega)$ and $u \in \mathcal{D}(U_{x_0} \cap \Omega, \mathbb{C}^N)$.

Suppose that $\mathcal{A}$ is not normally elliptic. Then there exist $x_0 \in \overline{\Omega}, \lambda_0 \in [\operatorname{Re} z \geq 0], \xi_0 \in \mathbb{R}^n \backslash \{0\}$, and $\eta \in \mathbb{C}^N \backslash \{0\}$ so that

$$(\lambda_0 + a_\pi(x_0, \xi_0))\eta = 0 \tag{2.19}$$

Choose any nonzero $\varphi \in \mathcal{D}(U \cap \Omega, \mathbb{R}^+)$, where $U := U_{x_0}$, and put

$$u_\tau(x) := \tau^{-2} e^{i\tau<x,\xi_0>} \varphi(x)\eta, \quad x \in \overline{\Omega}, \ \ \tau > 0 \ . \tag{2.20}$$

Then

$$\mathcal{A}_\pi(x_0, \partial)u_\tau = \tau^2 a_\pi(x_0, \xi_0)u_\tau + O(\tau^{-1}) \ , \tag{2.21}$$

$$\|u_\tau\|_p = O(\tau^{-2}) \ , \tag{2.22}$$

and there exists a constant $\gamma > 0$ so that

$$\|u_\tau\|_{2,p} \geq \gamma + O(\tau^{-1}) \tag{2.23}$$

for $\tau \to \infty$. If $\lambda_0 = 0$ we deduce from (2.18)-(2.23) that

$$\gamma \leq O(\tau^{-1}), \quad \tau \to \infty \ , \tag{2.24}$$

which is impossible. If $\lambda_0 \neq 0$ we obtain from (2.18)-(2.23) again the contradictory statement (2.24), since we can now use (2.18) with $\lambda := \tau^2\lambda_0$, provided $\tau^2 > \omega/|\lambda_0|$. This shows that $\mathcal{A}$ has to be normally elliptic. ∎

3. Linear reaction-diffusion systems. We shall now impose slightly stronger regularity conditions on the coefficients of $(\mathcal{A}, \mathcal{B})$, as well as an additional structural assumption. For this we introduce the Banach space

$$\mathbb{S}_p(\Omega) := H^1_{\hat{p}}(\Omega, X)^{n^2} \times H^1_{\hat{p}}(\Omega, X)^n \times L_{\hat{p}}(\Omega, X) \times B^{1-1/\hat{p}}_{\hat{p},\hat{p}}(\partial\Omega, X)$$

where $X := \mathcal{L}(\mathbb{K}^N)$. We denote the general point of $\mathbb{S}_p(\Omega)$ by

$$\sigma := ((a_{jk}), (a_1, \ldots a_n), a_0, b_0)$$

and define a map

$$s_p : \mathbb{S}_p(\Omega) \to \mathbb{E}_p(\Omega)$$

by setting

$$s_p(\sigma) := ((a_{jk}, (\hat{a}_1, \ldots, \hat{a}_n), a_0, (\hat{b}_1, \ldots, \hat{b}_n), b_0, 1) \ ,$$

where

$$\hat{a}_k := a_k - \partial_j a_{jk}, \quad \hat{b}_k := a_{jk}\nu^j, \quad k = 1, \ldots, n \ .$$

It is an easy consequence of Sobolev's imbedding theorem and the trace theorem that

$$s_p \in \mathcal{L}(\mathbb{S}_p(\Omega), \mathbb{E}_p(\Omega)) \ . \tag{3.1}$$

Hence

$$\mathcal{S}_p(\Omega) := s_p^{-1}(\mathcal{E}_p(\Omega)) \ \text{ is open in } \ \mathbb{S}_p(\Omega) \ . \tag{3.2}$$

Observe that $(\mathcal{A}_p(s_p(\sigma)), \mathcal{B}_p(s_p(\sigma)))$ is normally elliptic if and only if $\sigma \in \mathcal{S}_p(\Omega)$.

By abuse of notation we put

$$\mathcal{A} := \mathcal{A}(\sigma) := \mathcal{A}_p(s_p(\sigma)) = -\partial_j(a_{jk}\partial_k\cdot) + a_j\partial_j + a_0 \ ,$$

$$\mathcal{B} := \mathcal{B}(\sigma) := \mathcal{B}_p(s_p(\sigma)) = \delta(a_{jk}\nu^j\gamma\partial_k + b_0\gamma) + (1-\delta)\gamma$$

for $\sigma \in \mathcal{S}_p(\Omega)$ and call $(\mathcal{A}(\sigma), \mathcal{B}(\sigma))$ a *linear reaction-diffusion system (in* $\mathbf{L_p}$).

We denote by a^T the transposed of $a \in \mathcal{L}(\mathbb{C}^N)$. Then we define a map

$$s_p^\sharp : \mathbb{S}_p(\Omega) \to \mathbb{E}_p(\Omega)$$

by

$$s_p^\sharp(\sigma) := ((a_{jk}^\sharp), (a_1^\sharp, \dots, a_n^\sharp), a_0^\sharp, (b_1^\sharp, \dots, b_n^\sharp), b_0^\sharp, 1) \ ,$$

where

$$a_{jk}^\sharp := a_{kj}^T \ , \ a_j^\sharp := -\partial_k a_{jk}^T - a_j^T \ , \ a_0^\sharp := a_0^T - \partial_j a_j^T \ , \ b_j^\sharp := a_{kj}^T \nu^k \ , \ b_0^\sharp := a_j^T \nu^j + b_0^T \ .$$

It is not difficult to verify that

$$s_p^\sharp \in \mathcal{L}(\mathbb{S}_p(\Omega), \mathbb{E}_p(\Omega)) \ . \tag{3.3}$$

We put

$$\mathcal{A}^\sharp := \mathcal{A}^\sharp(\sigma) := \mathcal{A}_p(s_p^\sharp(\sigma)) = -\partial_j(a_{jk}^\sharp \partial_k \cdot) + a_j^\sharp \partial_j + a_0^\sharp$$

$$\mathcal{B}^\sharp := \mathcal{B}^\sharp(\sigma) := \mathcal{B}_p(s_p^\sharp(\sigma)) = \delta(a_{jk}^\sharp \nu^j \gamma \partial_k + b_0^\sharp \gamma) + (1-\delta)\gamma$$

and call $(\mathcal{A}^\sharp(\sigma), \mathcal{B}^\sharp(\sigma))$ the *formal adjoint* of the linear reaction-diffusion system $(\mathcal{A}(\sigma), \mathcal{B}(\sigma))$ if $\sigma \in \mathcal{S}_p(\Omega)$.

The following lemma shows that the formal adjoint of a linear reaction-diffusion system is normally elliptic.

Lemma 3.1. $s_p^\sharp(\mathcal{S}_p(\Omega)) \subset \mathcal{E}_p(\Omega)$.

Proof: We introduce two further boundary operators

$$\mathcal{C} := (\delta - 1) a_{jk} \nu^j \gamma \partial_k + \delta\gamma$$

and

$$\mathcal{C}^\sharp := (\delta - 1)(a_{jk}^\sharp \nu^j \gamma \partial_k + b_0^\sharp \gamma) + \delta\gamma.$$

Then it is easily verified that the following Green's formula

$$\langle v, \mathcal{A}u \rangle + \langle \mathcal{C}^\sharp v, \mathcal{B}u \rangle_\partial = \langle \mathcal{A}^\sharp v, u \rangle + \langle \mathcal{B}^\sharp v, \mathcal{C}u \rangle_\partial \tag{3.4}$$

is valid for $(v, u) \in H_{p'}^2 \times H_p^2$, where

$$\langle v, u \rangle := \int_\Omega \langle v(x), u(x) \rangle \, dx, \quad (v, u) \in L_{p'} \times L_p \tag{3.5}$$

and

$$\langle v, u \rangle_\partial := \int_{\partial\Omega} \langle v(x), u(x) \rangle \, d\sigma, \quad (v, u) \in L_{p'}(\partial\Omega, \mathbb{C}^N) \times L_p(\partial\Omega, \mathbb{C}^N) \ ,$$

with $\langle \xi, \eta \rangle := \xi^r \eta^r$ for $\xi, \eta \in \mathbb{C}^N$.

Given $\sigma \in \mathcal{E}_p(\Omega)$, it is obvious that $\mathcal{A}^\sharp(\sigma)$ is normally elliptic. The fact that $\mathcal{B}^\sharp(\sigma)$ satisfies the normal complementing condition with respect to $\mathcal{A}^\sharp(\sigma)$ follows from Green's formula, similarly as in the proof of [20, Teorema 3.1] (cf. also [46, Satz 14.5]). ∎

Finally we put

$$\pi_p := \pi \circ s_p \in \mathcal{L}(\mathbb{S}_p(\Omega), E(\Omega)) \ ,$$

so that

$$\pi_p(\sigma) := ((a_{jk}), (a_{jk}\nu^j)_{1\le k\le n}, 1)$$

for

$$\sigma := ((a_{jk}), (a_j)_{1\le j\le n}, a_0, b_0) \in \mathbb{S}_p(\Omega) \ .$$

Observe that

$$\mathcal{S}_p(\Omega) = \pi_p^{-1}(\mathcal{E}(\Omega)) \ . \tag{3.6}$$

In addition we have the following

Lemma 3.2. *π_p is a compact linear map.*

Proof: This follows from the compactness of the inclusion map $H^1_{\hat{p}}(\Omega, X) \hookrightarrow C(\overline{\Omega}, X)$ and $B^{1-1/\hat{p}}_{\hat{p},\hat{p}}(\partial\Omega, X) \hookrightarrow C(\partial\Omega, X)$, respectively. ∎

Corollary 3.3. *Let B be a bounded subset of $\mathcal{S}_p(\Omega)$. Then $\pi_p(B)$ is relatively compact in $\mathcal{E}(\Omega)$ if and only if $\pi_p(B)$ is bounded away from $\partial\mathcal{E}(\Omega)$.*

4. Examples. It is the purpose of this section to investigate the assumption that $(\mathcal{A}, \mathcal{B})$ be normally elliptic. For this we assume that $(\mathcal{A}, \mathcal{B})$ is given by (1.1), (1.2) and by (1.5) – (1.7).

Since the hypothesis that $\mathcal{A}$ be normally elliptic is a rather easy one, it remains to find easy conditions guaranteeing that $\mathcal{B}$ satisfies the normal complementing condition with respect to $\mathcal{A}$. This will not be possible, in general, without further restrictions on $\mathcal{A}$ since there exists a normally elliptic operator $\mathcal{A}$ so that the Dirichlet boundary operator does not satisfy the normal complementing condition with respect to $\mathcal{A}$ (cf. [26, p. 625]).

Given a continuous vector field $\beta \in C(\partial\Omega, \mathbb{R}^n)$ on $\partial\Omega$, we denote by ∂_β the directional derivative on $\partial\Omega$, that is,

$$\partial_\beta := \beta^j \gamma \partial_j \ .$$

Of course, β is 'outward pointing and nowhere tangent' if $(\beta(x)|\nu(x)) > 0$ for all $x \in \partial\Omega$, where $(\cdot\,|\,\cdot)$ is the euclidean inner product in $\mathbb{R}^n$.

Proposition 4.1. *Suppose that*

$$\sigma(a_\pi(x,\xi)) \subset [Re\, z > 0], \quad (x,\xi) \in T(\partial\Omega) \ , \ \xi \ne 0 \ ,$$

and that the Dirichlet operator, $\mathcal{B}_D := \gamma$, satisfies the normal complementing condition with respect to $\mathcal{A}$. Let β be a continuous, outward pointing, nowhere tangent vector field on $\partial\Omega$ and suppose that

$$\mathcal{B} = \delta(b\partial_\beta + b_0\gamma) + (1-\delta)c\gamma$$

with $b, c \in C(\partial\Omega, \mathcal{GL}(\mathbb{C}^N))$. *Finally suppose that* $\delta|\Gamma \in \{0,1\}$ *for every* $\Gamma \in \boldsymbol{\Gamma}$. *Then* $\mathcal{B}$ *satisfies the normal complementing condition with respect to* $\mathcal{A}$.

Proof: Fix $\Gamma \in \boldsymbol{\Gamma}$. Since condition (1.10) is purely local, we can assume without loss of generality that $\Gamma = \partial\Omega$.

If $\delta = 0$, then we can assume — by multiplying from the left by c^{-1} — that $\mathcal{B} = \mathcal{B}_D$ so that the assertion is true by assumption. If $\delta = 1$, then — by multiplying from the left by b^{-1} and by observing that condition (1.10) involves the principal boundary symbol only — we can assume that $\mathcal{B} = \partial_\beta$. Hence it suffices to consider the latter case.

We fix $(x, \xi) \in T(\partial\Omega)$ and $\lambda \in [\operatorname{Re} z \geq 0]$ with $(\xi, \lambda) \neq (0,0)$ arbitrarily and omit these quantities from the notation (whenever possible). The proof of Theorem 2.1 shows that the polynomial

$$z \mapsto \det(\lambda + a_\pi(x, \xi - \nu(x)z))$$

possesses precisely N roots $z_1, \ldots, z_N$ (counted according to their multiplicities) in $[\operatorname{Im} z > 0]$. Put

$$p(x) := \prod_{r=1}^{N} (z - z_r), \quad z \in \mathbb{C} .$$

Moreover, given $c \in \mathcal{GL}(\mathbb{C}^N)$, we denote by c^{ad} the 'algebraic adjoint' of c, that is, c^{ad} is the transposed of the matrix of cofactors of c, so that $c^{ad} = (\det c)c^{-1}$. (Of course, $c^{ad} := 1$ if $N = 1$.)

We define $Q := [Q_0, Q_1, \ldots, Q_{N-1}] \in \mathcal{L}(\mathbb{C}^{N^2}, \mathbb{C}^N)$ by

$$b_\pi(x, \xi - \nu(x)z)(\lambda + a_\pi(x, \xi - \nu(x)z))^{ad} = \mathcal{P}(z)p(z) + \sum_{j=0}^{N-1} Q_j z^j , \quad z \in \mathbb{C} , \tag{4.1}$$

where $\mathcal{P}$ is a suitable polynomial with coefficients in $\mathcal{L}(\mathbb{C}^N)$. Then it can be shown along the lines of the proof of [4, Part II, Theorem 3.2], for example, that (1.10) is equivalent to

$$Q \in \mathcal{SL}(\mathbb{C}^{N^2}, \mathbb{C}^N) \tag{4.2}$$

(for every possible value of (x, ξ, λ), of course).

Suppose that (4.2) is not true. Then there exists $\zeta \in \mathbb{C}^N \backslash \{0\}$ so that $\zeta \perp im\, Q$. Hence it follows from (4.1) that

$$\zeta^T b_\pi(x, \xi - \nu(x)z)(\lambda + a_\pi(x, \xi - \nu(x)z))^{ad} = p(z)\zeta^T \mathcal{P}(z), \quad z \in \mathbb{C} . \tag{4.3}$$

Observe that

$$b_\pi(x, \xi - \nu(x)z) = [(\beta|\xi) - (\beta|\nu)z]1 \in \mathcal{L}(\mathbb{C}^N), \quad z \in \mathbb{C} .$$

Consequently (4.3) takes the form

$$[(\beta|\xi) - (\beta|\nu)z]\zeta^T(\lambda + a_\pi(x, \xi - \nu(x)z))^{ad} = p(z)\zeta^T \mathcal{P}(z), \quad z \in \mathbb{C} . \tag{4.4}$$

Since p has no real roots, we see that (each element of the matrix) $\mathcal{P}(z)$ has to be divisible by $(\beta|\xi) - (\beta|\nu)z$. Hence there exists a polynomial $\tilde{\mathcal{P}}$ with coefficients in $\mathcal{L}(\mathbb{C}^N)$ so that

$$\zeta^T(\lambda + a_\pi(x, \xi - \nu(x)z))^{ad} = p(z)\zeta^T\tilde{\mathcal{P}}(z), \quad z \in \mathbb{C} . \tag{4.5}$$

This shows that the rows of the matrix

$$(\lambda + a_\pi(x, \xi - \nu(x)z))^{ad} ,$$

considered as polynomials in z, are linearly dependent modulo the polynomial p. By invoking again [4, Part II, Theorem 3.2] we deduce that the Dirichlet operator (for which $b_\pi(x, \xi - \nu(x)z) = 1 \in \mathcal{L}(\mathbb{C}^N)$) does not satisfy the normal complementing condition with respect to $\mathcal{A}$, which contradicts our hypothesis. This proves the assertion. ▮

Recall that (1.1) is *strongly uniformly elliptic* if it satisfies the *uniform Legendre-Hadamard condition*, that is,

$$\operatorname{Re} a_{jk}^{rs}(x)\xi^j\xi^k\eta_r\overline{\eta}_s > 0, \quad x \in \overline{\Omega}, \quad \xi \in \mathbb{R}^n\backslash\{0\},\ \eta \in \mathbb{C}^N\backslash\{0\} . \tag{4.6}$$

Theorem 4.2. *Suppose that $\mathcal{A}$ is strongly uniformly elliptic and that β is an outward pointing, nowhere vanishing, continuous vector field on $\partial\Omega$. Moreover suppose that $\delta|\Gamma \in \{0,1\}$ for every $\Gamma \in \Gamma$ and*

$$\mathcal{B} = \delta(b\partial_\beta + b_0\gamma) + (1-\delta)c\gamma$$

with $b, c \in C(\partial\Omega, \mathcal{GL}(\mathbb{C}^N))$. Then $(\mathcal{A}, \mathcal{B})$ is normally elliptic.

Proof: It is obvious that (4.6) implies that $\mathcal{A}$ is normally elliptic. Moreover it follows from [8, Lemma 6.3] that the Dirichlet boundary operator $\mathcal{B}_D$ satisfies the normal complementing condition with respect to $\mathcal{A}$. Hence the assertion is a consequence of Proposition 4.1. ▮

It should be remarked that the proof of Proposition 4.1 is simply an amplification of [8, Lemma 6.3].

If we restrict the class of differential operators on Ω further we can admit more general boundary operators. For this we recall that $\mathcal{A}$ is *very strongly uniformly elliptic* if it satisfies the *uniform Legendre condition*, that is, if

$$\operatorname{Re} a_{jk}^{rs}(x)\zeta_r^j\overline{\zeta_s^k} > 0, \quad x \in \overline{\Omega},\ \zeta \in \mathbb{C}^{nN}\backslash\{0\} . \tag{4.7}$$

It is obvious that

$$(4.7) \implies (4.6) \implies (1.4) .$$

Moreover, the converse implications hold if and only if $N = 1$.

Observe that the following result imposes no restriction upon δ.

Proposition 4.3. *Suppose that $\mathcal{A}$ is very strongly uniformly elliptic and*

$$\mathcal{B} = \delta(a_{jk}\nu^j\gamma\partial_k + b_0\gamma) + (1-\delta)\gamma \ .$$

Then $(\mathcal{A}, \mathcal{B})$ is normally elliptic.

Proof: This follows by modifying the proof of Lemma 6.5 in [8] (cf. [35, Appendix]), which is not conclusive, since it is based upon the false identiy (13) in Lemma 6.4 of [8]. ∎

We consider now the important special case of *separated divergence-form systems*, that is, we assume that

$$a_{jk} = \mathbf{A}\alpha_{jk}, \quad 1 \le j,k \le n \ , \tag{4.8}$$

where

$$\begin{gathered} \mathbf{A} \in H^1_{\hat{p}}(\Omega, \mathcal{L}(\mathbb{K}^N)), \quad \alpha := [\alpha_{jk}] \in H^1_{\hat{p}}(\Omega, \mathcal{L}(\mathbb{R}^n)), \\ \text{and } \alpha \text{ is symmetric and uniformly positive definite.} \end{gathered} \tag{4.9}$$

Moreover, we denote by

$$\nu_\alpha := \alpha\nu = (\alpha_{jk}\nu^k)_{1 \le j \le n}$$

the outer conormal with respect to α, and assume that

$$\mathcal{A} := -\partial_j(\mathbf{A}\alpha_{jk}\partial_k\cdot) + a_j\partial_j + a_0, \quad \mathcal{B} := \delta(\mathbf{A}\partial_{\nu_\alpha} + b_0\gamma) + (1-\delta)\gamma \ . \tag{4.10}$$

Then we have the following

Theorem 4.4. *Let conditions* (4.8) – (4.10) *be satisfied and suppose that*

$$\sigma(\mathbf{A}(x)) \subset [\operatorname{Re} z > 0], \quad x \in \overline{\Omega} \ . \tag{4.11}$$

Then $(\mathcal{A}, \mathcal{B})$ is normally elliptic, provided

$$(1-\delta(x))\mathbf{A}(x)\delta(x) = 0, \quad x \in \partial\Omega \ . \tag{4.12}$$

Proof: It is obvious that (4.8) – (4.11) imply that $\mathcal{A}$ is normally elliptic. We fix $x \in \partial\Omega$, omit it from the notation whenever possible, and put

$$\alpha(\xi) := \alpha_{jk}(x)\xi^j\xi^k, \quad \xi \in \mathbb{C}^n \ .$$

Then, letting $D := -i\partial_t$ and $\mathcal{B} := \mathcal{B}_D$, the boundary value problem (1.10) takes the form

$$[\lambda + \mathbf{A}\alpha(\xi - \nu D)]u = 0, \quad t > 0, \ u(0) = 0 \ , \tag{4.13}$$

where $\xi \in T_x(\partial\Omega)$ and $\lambda \in [\operatorname{Re} z \ge 0]$ with $(\xi, \lambda) \ne (0,0)$. There exists $S \in \mathcal{GL}(\mathbb{C}^N)$ so that $\mathbf{A} = S^{-1}JS$, where J is the Jordan normal form of $\mathbf{A}$. Hence, letting $v := Su$, we see that (4.13) is equivalent to

$$[\lambda + J\alpha(\xi - \nu D)]v = 0, \quad t > 0, \ v(0) = 0 \ ,$$

and v is exponentially decaying if and only if u has this property. Since J induces a direct sum decomposition of $\mathbb{C}^N$, according to the individual Jordan blocks, it suffices obviously to consider a single Jordan block. Hence it is enough to consider a system of the form

$$\begin{aligned} [\lambda + \mu\alpha(\xi - \nu D)]v_j + \alpha(\xi - \nu D)v_{j+1} = 0, \quad j = 1, \ldots, m-1\ , \\ [\lambda + \mu\alpha(\xi - \nu D)]v_m = 0 \end{aligned} \tag{4.14}$$

for $t > 0$, where $1 \leq m \leq N$ and $\mu \in [\operatorname{Re} z > 0]$ is an eigenvalue of $\mathbf{A}$, together with the initial conditions

$$v_1(0) = \ldots = v_m(0) = 0\ . \tag{4.15}$$

The characteristic polynomial of the last second order ordinary differential equation in (4.15) is apparently equivalent to the polynomial

$$z \mapsto \alpha(\xi - \nu z) + \lambda/\mu\ . \tag{4.16}$$

It is easily verified that $\lambda/\mu \in \mathbb{C}\backslash(-\infty, 0)$. Using this fact, together with $a(\xi - \nu z) > 0$ for $z \in \mathbb{R}\backslash\{0\}$, it follows that (4.16) has no real roots for $\xi \in \mathbb{R}^n$ and $\lambda \in [\operatorname{Re} z \geq 0]$ with $(\xi, \lambda) \neq (0, 0)$. Hence we deduce from Rouché's theorem that the number of roots of (4.16) in $[\pm \operatorname{Im} z > 0]$ is independent of (ξ, λ), hence equal to the corresponding number of the pair $(0, 1)$, that is, of the polynomial

$$z \mapsto \alpha(\nu) z^2 + \mu^{-1}\ . \tag{4.17}$$

Since $\alpha(\nu) > 0$ and $\arg \mu \in (-\pi/2, \pi/2)$, it is easily seen that (4.17) possesses precisely one root in $[\operatorname{Im} z > 0]$. Hence (4.16) has precisely one root in $[\operatorname{Im} z > 0]$. This implies that the last differential equation in (4.14) has precisely one linearly independent exponentially decaying solution of the form

$$v_m(t) = e^{-\varepsilon t} v_m(0), \quad t > 0\ ,$$

for some $\varepsilon > 0$. Consequently $v_m = 0$, thanks to (4.15). Now it follows from (4.14) and (4.15) that

$$[\lambda + \mu\alpha(\xi - \nu D)]v_{m-1} = 0, \quad t > 0,\ v_{m-1}(0) = 0\ ,$$

which implies — by what has just been shown — that $v_{m-1} = 0$. By induction we see finally that $v_1 = \ldots = v_m = 0$. This proves that the Dirichlet boundary operator satisfies the normal complementing condition with respect to $\mathcal{A}$.

Observe that $\delta(x)$ is an orthogonal projection in $\mathcal{L}(\mathbb{C}^N)$. Hence $\mathbb{C}^N = V \oplus W$ with $V := \delta(x)\mathbb{C}^N$ and $W := (1 - \delta(x))\mathbb{C}^N$. With respect to this direct sum decomposition $\mathbf{A}(x)$ has, thanks to (4.12), the matrix representation

$$\mathbf{A}(x) = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ 0 & \mathbf{A}_{22} \end{bmatrix}\ . \tag{4.18}$$

Thus the boundary value problem (1.10) decomposes into the system

$$[\lambda + \mathbf{A}_{11}\alpha(\xi - \nu D)]v + \mathbf{A}_{12}\alpha(\xi - \nu D)w = 0, \quad [\lambda + \mathbf{A}_{22}\alpha(\xi - \nu D)]w = 0 \tag{4.19}$$

for $t > 0$ and

$$\mathbf{A}_{11}\beta(\xi - \nu D)v(0) + \mathbf{A}_{12}\beta(\xi - \nu D)w(0) = 0, \quad w(0) = 0, \tag{4.20}$$

where $\beta(\xi - \nu D) := (\nu_a|\xi) - (\nu_\alpha|\nu)D$.

It follows from (4.18) that

$$\sigma(\mathbf{A}(x)) = \sigma(\mathbf{A}_{11}) \cup \sigma(\mathbf{A}_{22}) \ . \tag{4.21}$$

Hence we can apply the first part of the proof (replacing $\mathbb{C}^N$ by V) to the second equations of (4.19) and (4.20), respectively, to deduce that $w = 0$. Thus we are left with the boundary value problem

$$[\lambda + \mathbf{A}_{11}\alpha(\xi - \nu D)]v = 0, \quad t > 0,\ \beta(\xi - \nu D)v(0) = 0 \ . \tag{4.22}$$

Since we can assume for this part of the proof without loss of generality that $\Gamma = \partial\Omega$, we deduce from (4.21), (4.11), the first part of the proof, and Proposition 4.1 that $v = 0$, since the boundary condition in (4.22) corresponds to the boundary operator $\mathcal{B} := \partial_{\nu_\alpha}$ (with $\mathbb{C}^N$ replaced by V, of course). This proves the theorem. ∎

Examples 4.5. (a) Suppose that $N = 2$ and (4.8)−(4.10) are satisfied with $\mathbb{K} = \mathbb{R}$. Then condition (4.11) is fulfilled if and only if

$$\det \mathbf{A}(x) > 0 \ \text{ and } \ tr\,\mathbf{A}(x) > 0,\ x \in \overline{\Omega} \ .$$

This follows immediately from the fact that the determinant is the product and the trace is the sum of the eigenvalues and that complex eigenvalues occur in conjugate complex pairs.

(b) Condition (4.12) is automatically satisfied if either $\delta|\Gamma \in \{0, 1\}$ for each $\Gamma \in \boldsymbol{\Gamma}$ or $\mathbf{A}(x)$ is a diagonal matrix for each $x \in \partial\Omega$. In general, condition (4.12) is satisfied if and only if there exist for each $\Gamma \in \boldsymbol{\Gamma}$ permutations of the rows and of the columns of $\mathbf{A}(x), x \in \Gamma$, so that, *after applying these permutations*, $\mathbf{A}(x)$ has the block triangular representation

$$\mathbf{A}(x) = \begin{bmatrix} \mathbf{A}_{11}(x) & \mathbf{A}_{12}(x) \\ 0 & \mathbf{A}_{22}(x) \end{bmatrix} \in \mathcal{L}(\mathbb{K}^{N_1} \times \mathbb{K}^{N_2})$$

and

$$\delta(x) = \operatorname{diag}(1, 0) \in \mathcal{L}(\mathbb{K}^{N_1} \times \mathbb{K}^{N_2})$$

for $x \in \Gamma$, where $N_1 + N_2 = N$.

(c) Suppose that $\alpha = 1$, that is, $\alpha_{jk} = \delta_{jk}$, the Kronecker symbol. Then $\mathcal{A}$ is very strongly uniformly elliptic if and only if the symmetric part $(\mathbf{A}(x) + \overline{\mathbf{A}^T(x)})/2$ of $\mathbf{A}(x)$ is positive definite for each $x \in \overline{\Omega}$. Hence the normal ellipticity of $(\mathcal{A}, \mathcal{B})$ follows in this case from Proposition 4.3 without any restriction upon δ. ∎

We close this section with a simple generalization of the above results.

Theorem 4.6. *Let conditions* (4.8) – (4.10) *be satisfied and suppose that* $\mathbf{A}$ *has block triangular structure, that is,*

$$\mathbf{A} = [\mathbf{A}_{hi}]_{1\le h,i\le m} \in H^1_{\hat{p}}(\Omega, \mathcal{L}(\mathbb{K}^{N_1} \times \ldots \times \mathbb{K}^{N_m}))$$

with $\mathbf{A}_{hi} = 0$ *for* $h > i$. *Moreover suppose that each one of the boundary value problems*

$$(\mathcal{A}_i, \mathcal{B}_i) := (-\partial_j(\mathbf{A}_{ii}\alpha_{jk}\partial_k\cdot),\quad \delta_i\mathbf{A}_{ii}\partial_{\nu_\alpha} + (1-\delta_i)\gamma),\quad 1 \le i \le m\ ,$$

(no summation with respect to i*), where* $\delta = \mathrm{diag}[\delta_1, \ldots, \delta_m]$ *corresponding to the decomposition* $\mathbb{K}^N = \mathbb{K}^{N_1} \times \ldots \times \mathbb{K}^{N_m}$, *is normally elliptic. Then* $(\mathcal{A}, \mathcal{B})$ *is normally elliptic.*

Proof: It suffices to observe that $\mathcal{A}$ is obviously normally elliptic, that (1.10) has now also block triangular structure, and that this system can be solved 'from the bottom,' yielding only zero as exponentially decaying solution, thanks to the normal ellipticity of the 'diagonal blocks.' ∎

Remark 4.7 Suppose that $(\mathcal{A}, \mathcal{B})$ is a separated divergence form system so that a_j, $a_0 \in L_{\hat{p}}(\Omega, \mathcal{L}(\mathbb{K}^N))$ and $b_0 \in B^{1-1/\hat{p}}_{\hat{p},\hat{p}}(\partial\Omega, \mathcal{L}(\mathbb{K}^N))$. Then condition (4.11) is necessary for $A \in \mathcal{H}(L_p)$, where A is the L_p-realization of $(\mathcal{A}, \mathcal{B})$.

This is a consequence of Theorem 2.4 and the obvious fact that $\mathcal{A}$ is normally elliptic if and only if (4.11) is satisfied.

Part Two: Technical preliminaries.

5. The extrapolation setting. For simplicity we assume from now on that

$$\Omega \text{ is of class } C^\infty\ .$$

Given $\sigma \in \mathcal{S}_p(\Omega)$, we define closed linear subspaces $H^s_{p,\mathcal{B}(\sigma)}$ of H^s_p for $s \in [0,2]\backslash(\mathbb{N}+1/p)$ by

$$H^s_{p,\mathcal{B}(\sigma)} := \begin{cases} \{u \in H^s_p\ ;\ \mathcal{B}(\sigma)u = 0\}, & 1+1/p < s \le 2\ , \\ \{u \in H^s_p\ ;\ (1-\delta)\gamma u = 0\}, & 1/p < s < 1+1/p\ , \\ H^s_p, & 0 \le s < 1/p\ . \end{cases}$$

A similar definition holds for $H^s_{p',\mathcal{B}^\sharp(\sigma)}$, $s \in [0,2]\backslash(\mathbb{N}+1/p')$. Then we define $H^s_{p,\mathcal{B}(\sigma)}$ for $s \in [-2,0)\backslash(\mathbb{Z}+1/p)$ by

$$H^s_{p,\mathcal{B}(\sigma)} := (H^{-s}_{p',\mathcal{B}^\sharp(\sigma)})' \tag{5.1}$$

with respect to the duality pairing induced by (3.5). Observe that $H^s_{p,\mathcal{B}(\sigma)}$ is independent of $\sigma \in \mathcal{S}_p(\Omega)$ for $s \in (-2+1/p, 1+1/p)\backslash(\mathbb{Z}+1/p)$. Hence we put

$$H^s_{p,\mathcal{B}} := H^s_{p,\mathcal{B}(\sigma)},\quad s \in (-2+1/p, 1+1/p)\backslash(\mathbb{Z}+1/p)\ . \tag{5.2}$$

In the following we denote by $[\cdot,\cdot]_\theta$, $0 < \theta < 1$, the standard complex interpolation functor (cf. [43, 16]), and we write $E \doteq F$ if E and F are Banach spaces differing by equivalent norms only. (If $\mathbb{K} = \mathbb{R}$ we have to complexify the spaces to use $[\cdot,\cdot]_\theta$. Then we get back to the real situation by setting $[E,F]_\theta := [E_\mathbb{C}, F_\mathbb{C}]_\theta \cap (E+F)$, equipped with the topology induced by $[E_\mathbb{C}, F_\mathbb{C}]_\theta$.)

Lemma 5.1. *$[L_p, H^2_{p,\mathcal{B}(\sigma)}]_\theta \doteq H^{2\theta}_{p,\mathcal{B}(\sigma)}$ and $[H^{-2}_{p,\mathcal{B}(\sigma)}, L_p]_{1-\theta} \doteq H^{-2\theta}_{p,\mathcal{B}(\sigma)}$, whenever the right hand sides are defined.*

Proof: It follows from the proof of Theorem 2.1 that

$$[\delta a_{jk}\nu^j\nu^k, (1-\delta)](x) \in \mathcal{SL}(\mathbb{K}^N \times \mathbb{K}^N, \mathbb{K}^N), \quad x \in \partial\Omega .$$

This implies that the boundary operator $\mathcal{B}(\sigma)$ is 'normal' in the sense of [37, Definition 3.1]. Hence the first assertion follows from [37, Theorem 4.1], since it is easily verified that our regularity conditions suffice for carrying through the proof of the latter theorem.

Lemma 3.1 implies that $\mathcal{B}^\sharp(\sigma)$ is also 'normal'. Thus $[L_{p'}, H^2_{p',\mathcal{B}^\sharp(\sigma)}]_\theta \doteq H^{2\theta}_{p',\mathcal{B}^\sharp(\sigma)}$ for $2\theta \in [0,2]\backslash(\mathbb{N} + 1/p')$. It is clear that

$$H^2_{p',\mathcal{B}^\sharp(\sigma)} \overset{d}{\hookrightarrow} L_{p'} ,$$

where $\overset{d}{\hookrightarrow}$ means 'dense injection,' and that these spaces are reflexive. Hence

$$L_p \overset{d}{\hookrightarrow} H^{-2}_{p,\mathcal{B}(\sigma)}$$

by (5.1) and the Hahn-Banach theorem. Now the second assertion follows from the duality theorem for the complex interpolation functor (e.g. [16, Corollary 4.5.2 and Theorem 4.2.1 (a)]). ∎

We denote now by

$$A(\sigma) \ \text{ the } L_p\text{-realization of } \ (\mathcal{A}(\sigma), \mathcal{B}(\sigma))$$

and by

$$A^\sharp(\sigma) \ \text{ the } L_{p'}\text{-realization of } \ (\mathcal{A}^\sharp(\sigma), \mathcal{B}^\sharp(\sigma)) .$$

Theorem 5.2. *Let B be a bounded subset of $\mathcal{S}_p(\Omega)$ so that $\pi_p(B)$ is bounded away from $\partial\mathcal{E}(\Omega)$. Then there exist a neighbourhood U of B in $\mathbb{S}_p(\Omega)$, a number $\vartheta \in (0, \pi/2)$, and positive constants κ and ω so that*

(i) $\lambda + A(\sigma) \in Isom(H^2_{p,\mathcal{B}(\sigma)}, L_p)$,

(ii) $\lambda + A^\sharp(\sigma) \in Isom(H^2_{p',\mathcal{B}^\sharp(\sigma)}, L_{p'})$,

(iii) $\kappa^{-1} \le \frac{\|(\lambda + A(\sigma))u\|_p}{|\lambda|\|u\|_p + \|u\|_{2,p}} \le \kappa$,

(iv) $\kappa^{-1} \le \frac{\|(\lambda + A^\sharp(\sigma))v\|_{p'}}{|\lambda|\|v\|_{p'} + \|v\|_{2,p'}} \le \kappa$

for $\sigma \in U, \lambda \in S(\vartheta, \omega), u \in H^2_{p,\mathcal{B}(\sigma)}\backslash\{0\}$ and $v \in H^2_{p',\mathcal{B}^\sharp(\sigma)}\backslash\{0\}$.

Proof: This is an easy consequence of (3.6), Theorem 2.3, and Corollary 3.3. ∎

Corollary 5.3. *$A(\sigma) \in \mathcal{H}(L_p)$ and $A^\sharp(\sigma) \in \mathcal{H}(L_{p'})$.*

It is an immediate consequence of Green's formula (3.4) that $A'(\sigma) \supset A^\sharp(\sigma)$. On the other hand it follows from Theorem 5.2 that $\lambda + A'(\sigma) = (\lambda + A(\sigma))'$ and $\lambda + A^\sharp(\sigma)$ are both bijective for an appropriately chosen $\lambda > 0$. Hence $A'(\sigma)$ cannot be a proper extension of $A^\sharp(\sigma)$, which proves

$$A^\sharp(\sigma) = A'(\sigma) := (A(\sigma))', \quad \sigma \in \mathcal{S}_p(\Omega) . \tag{5.3}$$

We write $A \in \mathcal{G}(X, M, \omega)$ if $-A$ is the infinitesimal generator of a strongly continuous semigroup $\{e^{-tA}\,; t \geq 0\}$ on the Banach space X so that $\|e^{-tA}\| \leq Me^{\omega t}$ for $t \geq 0$. Recall that the *type of* $-\boldsymbol{A}$, *type*$(-A)$, is the infimum of all $\omega \in \mathbb{R}$ so that $A \in \mathcal{G}(X, M, \omega)$ for some $M \geq 1$, and that $\mathrm{Re}[\, z \geq \omega] \subset \rho(-A)$ if $\omega >$ type$(-A)$.

We fix now any $\omega >$type$(-A(\sigma))$ and put

$$\begin{aligned} E &:= E_0 := (E, \|\cdot\|) := L_p \ , \\ E_1(\sigma) &:= (D(A(\sigma)), \|(\omega + A(\sigma)) \cdot \|) \ , \\ E_{-1}(\sigma) &:= (E, \|(\omega + A(\sigma))^{-1} \cdot \|)^\sim \ , \end{aligned}$$

where $\sim$ denotes 'completion'. Moreover,

$$E_\alpha(\sigma) := \begin{cases} [E, E_1(\sigma)]_\alpha, & 0 < \alpha < 1 \ , \\ [E_{-1}(\sigma), E]_{1+\alpha}, & -1 < \alpha < 0 \ . \end{cases}$$

It is easily seen that different choices of ω lead to the same spaces, except for equivalent norms.

Proposition 5.4. *$E_\alpha(\sigma) \doteq H^{2\alpha}_{p,\mathcal{B}(\sigma)}$ for $2\alpha \in [-2, 2] \backslash (\mathbb{Z} + 1/p)$.*

Proof: This is a consequence of Lemma 5.1, Corollary 5.3, formula (5.1), and [12, Theorem 1.3]. ∎

It follows from the last proposition that $E_\alpha(\sigma)$ is — as a vector space — independent of $\sigma \in \mathcal{S}_p(\Omega)$ for $2\alpha \in (-2 + 1/p, 1 + 1/p) \backslash (\mathbb{Z} + 1/p)$. Hence we put

$$E_\alpha := H^{2\alpha}_{p,\mathcal{B}}, \quad 2\alpha \in (-2 + 1/p, 1 + 1/p) \backslash (\mathbb{Z} + 1/p) \ , \tag{5.4}$$

and denote the norm of E_α by $\|\cdot\|_\alpha$. Since $\alpha < 1$, confusion with the L_p-norm is not possible.

The next proposition shows that the spaces E_α are stable with respect to complex interpolation.

Proposition 5.5. *$[E_\alpha, E_\beta]_\theta \doteq E_{(1-\theta)\alpha + \theta\beta}$ for $-2 + 1/p < 2\alpha < 2\beta < 1 + 1/p$ with $2\alpha, 2\beta, 2[(1 - \theta)\alpha + \theta\beta] \notin \mathbb{Z} + 1/p$.*

Proof: Put $\sigma_0 := ((\delta_{jk}1), 0, 0, 0) \in \mathcal{S}_p(\Omega)$, so that $\mathcal{A}(\sigma_0)$ is the negative diagonal Laplacian $-\Delta$ and $\mathcal{B}(\sigma_0) = \delta\frac{\partial}{\partial\nu} + (1 - \delta)\gamma$. Then it is known (e.g. [36, 18, 19]) that the purely imaginary powers $A^{it}(\sigma_0)$ of $A(\sigma_0)$ are uniformly bounded for t in a neighbourhood of 0 in $\mathbb{R}$. Hence the assertion follows from [9, Theorem 7]. ∎

We denote by

$$A_{\alpha-1}(\sigma) \ \text{ the closure of } A(\sigma) \text{ in } E_{\alpha-1}(\sigma) \text{ for } \alpha \in [0, 1].$$

Then we know by [9, Theorem 6] that $A_{\alpha-1}(\sigma) \in \mathcal{H}(E_{\alpha-1}(\sigma))$. The following theorem contains a somewhat more precise information for certain values of α.

Theorem 5.6. *Suppose that S is a nonempty subset of $\mathcal{S}_p(\Omega)$ and that there exist constants $\vartheta \in (0, \pi/2)$ and $\kappa, \omega \in (0, \infty)$ so that $\lambda + A(\sigma) \in Isom(E_1(\sigma), E)$,*

$$\kappa^{-1} \leq \frac{\|(\lambda + A(\sigma))u\|_p}{|\lambda|\|u\|_p + \|u\|_{2,p}} \leq \kappa \ , \tag{5.5}$$

and

$$\kappa^{-1} \leq \frac{\|(\lambda + A^\sharp(\sigma))v\|_{p'}}{|\lambda|\|v\|_{p'} + \|v\|_{2,p'}} \leq \kappa \tag{5.6}$$

for $\sigma \in S$, $\lambda \in S(\vartheta, \omega)$, $u \in E_1(\sigma)\backslash\{0\}$ and $v \in E_1^\sharp(\sigma)\backslash\{0\}$. Then, given α with $1/p < 2\alpha < 1 + 1/p$, there exists a positive constant $\tilde{\kappa} := \tilde{\kappa}(\alpha, \kappa, \omega)$ so that

$$\lambda + A_{\alpha-1}(\sigma) \in Isom(E_\alpha, E_{\alpha-1})$$

and

$$\tilde{\kappa}^{-1} \leq \frac{\|(\lambda + A_{\alpha-1}(\sigma))u\|_{\alpha-1}}{|\lambda|\|u\|_{\alpha-1} + \|u\|_\alpha} \leq \tilde{\kappa}$$

for $\sigma \in S$, $\lambda \in S(\vartheta, \omega)$ and $u \in E_\alpha\backslash\{0\}$.

Proof: It follows from (5.5) that

$$\kappa^{-1}\|u\|_{2,p} \leq \|u\|_{E_1(\sigma)} \leq \kappa(1+\omega)\|u\|_{2,p}, \quad u \in E_1(\sigma), \quad \sigma \in S, \tag{5.7}$$

and (5.3) and (5.6) imply

$$\kappa^{-1}\|v\|_{2,p'} \leq \|v\|_{E_1^\sharp(\sigma)} \leq \kappa(1+\omega)\|v\|_{2,p'}, \quad v \in E_1^\sharp(\sigma), \quad \sigma \in S, \tag{5.8}$$

where

$$E_1^\sharp(\sigma) := (D(A^\sharp(\sigma)), \|(\omega + A^\sharp(\sigma)) \cdot \|_{p'}) \ .$$

Hence, putting

$$E_\alpha^\sharp(\sigma) := [E^\sharp, E_1^\sharp(\sigma)]_\alpha, \ 0 < \alpha < 1, \ \sigma \in S, \tag{5.9}$$

where $E^\sharp := E_0^\sharp := L_{p'}$, we obtain from (5.7) and (5.8) by interpolation (since $id \in Isom(E_1(\sigma), H^2_{p,\mathcal{B}(\sigma)})$ and $id \in Isom(E_1^\sharp(\sigma), H^2_{p',\mathcal{B}^\sharp(\sigma)})$) and density that

$$\kappa^{-\alpha}\|u\|_{2\alpha,p} \leq \|u\|_{E_\alpha(\sigma)} \leq [\kappa(1+\omega)]^\alpha\|u\|_{2\alpha,p}, \quad u \in E_\alpha(\sigma), \quad \sigma \in S, \tag{5.10}$$

and

$$\kappa^{-\beta}\|v\|_{2\beta,p'} \leq \|v\|_{E_\beta^\sharp(\sigma)} \leq [\kappa(1+\omega)]^\beta\|v\|_{2\beta,p'}, \quad v \in E_\beta^\sharp(\sigma), \ \sigma \in S, \tag{5.11}$$

for $2\alpha \in [0,2]\backslash(\mathbb{N}+1/p)$ and $2\beta \in [0,2]\backslash(\mathbb{N}+1/p')$, respectively. From [9, Theorem 11] we know that $E_{-\alpha}(\sigma) = (E_\alpha^\sharp(\sigma))'$, $0 \leq \alpha \leq 1$, $\sigma \in S$, with respect to the duality pairing induced by (3.5). Hence it follows from (5.10), (5.11), and the definition of the dual norm that

$$[\kappa(1+\omega)]^{-|\alpha|}\|u\|_\alpha \leq \|u\|_{E_\alpha(\sigma)} \leq [\kappa(1+\omega)]^{|\alpha|}\|u\|_\alpha, \quad u \in E_\alpha, \ \sigma \in S, \tag{5.12}$$

for $2\alpha \in (-2+1/p, 1+1/p)\backslash(\mathbb{Z}+1/p)$.

In the following we denote by c positive constants, which may be different from expression to expression and may depend upon κ, ω and α, but are always independent of $\sigma \in S$ and $\lambda \in S(\vartheta, \omega)$.

It follows from (5.5) that

$$\begin{aligned} \|(\lambda + A(\sigma))^{-1}u\|_{E_{-1}(\sigma)} &= \|(\omega + A(\sigma))^{-1}(\lambda + A(\sigma))^{-1}u\| \\ &= \|(\lambda + A(\sigma))^{-1}(\omega + A(\sigma))^{-1}u\| \\ &\le \|(\lambda + A(\sigma))^{-1}\|_{\mathcal{L}(E)}\|u\|_{E_{-1}(\sigma)} \\ &\le \frac{\kappa}{|\lambda|}\|u\|_{E_{-1}(\sigma)} \end{aligned}$$

for $u \in E_1(\sigma)$ and $\lambda \in S(\vartheta, \omega)$. Thus interpolation, (5.12), and a density argument give

$$\|(\lambda + A_{\alpha-1}(\sigma))^{-1}u\|_{\alpha-1} \le \frac{c}{|\lambda|}\|u\|_{\alpha-1}, \quad u \in E_{\alpha-1}, \tag{5.13}$$

for $\sigma \in S$ and $1/p < 2\alpha < 1 + 1/p$.

Using the identity

$$(\lambda + A(\sigma))^{-1}(\omega + A(\sigma)) = (\omega - \lambda)(\lambda + A(\sigma))^{-1} + 1 ,$$

we obtain from (5.5) that

$$\begin{aligned} \|(\lambda + A(\sigma))^{-1}u\| &= \|(\lambda + A(\sigma))^{-1}(\omega + A(\sigma))(\omega + A(\sigma))^{-1}u\| \\ &\le |\omega - \lambda|\|(\lambda + A(\sigma))^{-1}(\omega + A(\sigma))^{-1}u\| + \|u\|_{E_{-1}(\sigma)} \\ &\le c(\frac{\omega + |\lambda|}{|\lambda|} + 1)\|u\|_{E_{-1}(\sigma)} \end{aligned} \tag{5.14}$$

$$\le c\|u\|_{E_{-1}(\sigma)} ,$$

whereas (5.5) implies directly

$$\|(\lambda + A(\sigma))^{-1}u\|_{E_1(\sigma)} \le \kappa\|u\|$$

for $\sigma \in S, \lambda \in S(\vartheta, \omega)$ and $u \in E_1(\sigma)$. Thus, by interpolation, by (5.12), and by a density argument,

$$\|(\lambda + A_{\alpha-1}(\sigma))^{-1}u\|_{\alpha} \le c\|u\|_{\alpha-1}, \quad u \in E_{\alpha-1} , \tag{5.15}$$

for $\sigma \in S$, $\lambda \in S(\vartheta, \omega)$ and $1/p < 2\alpha < 1 + 1/p$.

Finally we deduce from (5.5) and (5.10) that

$$\begin{aligned} \|A(\sigma)u\|_{E_{-1}(\sigma)} = \|(\omega + A(\sigma))^{-1}A(\sigma)u\| &= \|A(\sigma)(\omega + A(\sigma))^{-1}u\| \\ \le c\|(\omega + A(\sigma))^{-1}u\|_{E_1(\sigma)} &\le c\|u\| \end{aligned}$$

for $u \in E_1(\sigma)$ and $\sigma \in S$. Hence, again by interpolation, (5.12), and density arguments,

$$\|A_{\alpha-1}(\sigma)u\|_{\alpha-1} \le c\|u\|_{\alpha}, \quad u \in E_{\alpha} , \tag{5.16}$$

for $\sigma \in S$ and $1/p < 2\alpha < 1 + 1/p$. Now the assertion is an easy consequence of (5.13), (5.15), and (5.16). ∎

Corollary 5.7. *Let the hypotheses of Theorem 5.6 be satisfied and suppose that* $1 + 1/p < 2\alpha \leq 2$. *Then*

$$\lambda + A_{\alpha-1}(\sigma) \in Isom(E_\alpha(\sigma), E_{\alpha-1}), \quad \sigma \in S,\ \lambda \in S(\vartheta, \omega)\ ,$$

and there exists a constant $\tilde{\kappa} := \tilde{\kappa}(\alpha) > 0$ *so that*

$$\tilde{\kappa}^{-1}\|u\|_{2\alpha,p} \leq \|(\omega + A_{\alpha-1}(\sigma))u\|_{\alpha-1} \leq \tilde{\kappa}\|u\|_{2\alpha,p}, \quad u \in E_\alpha(\sigma),\ \sigma \in S\ . \tag{5.17}$$

Proof: This is a consequence of (5.10) and the fact that $\omega + A_{\alpha-1}(\sigma)$ is a norm isomorphism from $E_\alpha(\sigma)$ onto $E_{\alpha-1}(\sigma)$ (cf. [13, Theorem 6.1 (iii)]). ∎

It follows from (5.17) that the graph norms of $E_\alpha(\sigma)$ are equivalent for different $\sigma \in S$, uniformly with respect to $\sigma \in S$, although the spaces $E_\alpha(\sigma)$ and $E_\alpha(\sigma')$ are distinct as vector spaces, in general, if $\sigma \neq \sigma'$.

6. The Dirichlet form. Let X and Y be Banach spaces. Then we denote by $\mathcal{L}^2(X \times Y, \mathbb{K})$ the Banach space of all continuous bilinear forms $a : X \times Y \to \mathbb{K}$, equipped with the norm

$$\|a\| := \sup\{|a(x,y)|\ ;\ \|x\| \leq 1,\ \|y\| \leq 1\}\ .$$

Given $\sigma \in \mathbb{S}_p(\Omega)$, we define the *Dirichlet form of* $(\mathcal{A}(\sigma), \mathcal{B}(\sigma))$ by

$$a(\sigma)(v,u) := \langle \partial_j v, a_{jk}\partial_k u\rangle + \langle v, a_j\partial_j u + a_0 u\rangle + \langle \gamma v, b_0\gamma u\rangle_\partial$$

for $(v,u) \in H^2_{p'} \times H^2_p$. It is easily verified that

$$[\sigma \mapsto a(\sigma)] \in \mathcal{L}(\mathbb{S}_p(\Omega), \mathcal{L}^2(H^2_{p'} \times H^2_p, \mathbb{K}))$$

and that

$$a(\sigma)(v,u) = \langle v, A(\sigma)u\rangle\ , \quad (v,u) \in H^2_{p',\mathcal{B}^\sharp(\sigma)} \times H^2_{p,\mathcal{B}(\sigma)}\ . \tag{6.1}$$

We shall now show that $a(\sigma)$ has a continuous extension — denoted again by $a(\sigma)$ — to $(E_{\alpha-1})' \times E_\alpha$ for all α in a suitable neighbourhood of $1/2$. For this we observe first that, due to Sobolev's imbedding theorem and Hölder's inequality,

$$\mathbb{S}_p(\Omega) \hookrightarrow C^\rho(\overline{\Omega}, X)^{n^2} \times L_{\hat{p}}(\Omega, X)^n \times L_{\hat{p}}(\Omega, X) \times L_{\hat{p}}(\partial\Omega, X) \tag{6.2}$$

for $0 \leq \rho \leq 1 - n/\hat{p}$, where $X := \mathcal{L}(\mathbb{K}^N)$. In the following we denote by

$\mathbb{S}^\rho_p(\Omega)$, $0 \leq \rho \leq 1 - n/\hat{p}$, the vector space $\mathbb{S}_p(\Omega)$, equipped with the topology induced by the Banach space on the right hand side of (6.2).

Moreover, we put

$$\rho(\alpha) := \begin{cases} 0 & \text{if } 2\alpha = 1\ , \\ \rho & \text{if } 2\alpha \neq 1\ , \end{cases}$$

where ρ is arbitrarily fixed so that

$$|2\alpha - 1| < \rho \leq 1 - n/\hat{p}$$

and so that $\rho(\cdot)$ is an increasing function of $|2\alpha - 1|$. Observe that

$$\mathbb{S}^{\rho(\alpha)}_p(\Omega) \hookrightarrow \mathbb{S}^{\rho(\beta)}_p(\Omega), \quad \alpha \geq \beta,\ 1 \leq 2\alpha,\ 2\beta < (1 + 1/p) \wedge (2 - n/\hat{p})\ .$$

We can now prove an important continuity

Theorem 6.1. *Suppose that* $1 \leq 2\alpha < (1 + 1/p) \wedge (2 - n/\hat{p})$. *Then*

$$[\sigma \mapsto a(\sigma)] \in \mathcal{L}(\mathbb{S}_p^{\rho(\alpha)}(\Omega), \mathcal{L}^2((E_{\alpha-1})' \times E_\alpha))$$

and

$$a(\sigma)(v, u) = \langle v, A_{\alpha-1}(\sigma)u \rangle$$

for $(v, u) \in (E_{\alpha-1})' \times E_\alpha$ *and* $\sigma \in \mathcal{S}_p(\Omega)$.

Proof: Suppose that $1 \leq s < 1 + 1/p$. Then $H_{p'}^{1-s} \doteq (H_p^{s-1})'$, since $0 \leq s - 1 < 1/p$ (e.g. [43, Theorem 4.8.2]). Hence it follows that

$$\partial_j \in \mathcal{L}(H_{p'}^{2-s}, (H_p^{s-1})') \cap \mathcal{L}(H_p^s, H_p^{s-1}) \ .$$

Suppose that $0 < t < \rho < 1$ and $m \in C^\rho(\overline{\Omega}, \mathcal{L}(\mathbb{K}^N))$. Then it is a consequence of Strichartz' multiplier theorem ([42] cf. also [44, Theorem 3.3.2], [39, 30]) that

$$[u \mapsto mu] \in \mathcal{L}(H_p^t) \ ,$$

and

$$\|mu\|_{t,p} \leq c \|m\|_{C^\rho} \|u\|_{t,p} \quad , \quad u \in H_p^t \ .$$

Of course, this is also true if $t = \rho = 0$. Using these facts we deduce the first assertion easily from Sobolev type imbedding theorems, the trace theorem, and Hölder's inequality.

The second assertion follows now from (6.1) and the density of $E_1^\sharp(\sigma) \times E_1(\sigma)$ in $E_{1-\alpha}^\sharp \times E_\alpha = (E_{\alpha-1})' \times E_\alpha$. ▮

We denote by

$\mathcal{S}_p^{\rho(\alpha)}(\Omega)$, $1 \leq 2\alpha < (1 + 1/p) \wedge (2 - n/\hat{p})$, the set $\mathcal{S}_p(\Omega)$,

equipped with the topology of $\mathbb{S}_p^{\rho(\alpha)}(\Omega)$.

Lemma 6.2. $\mathcal{S}_p^{\rho(\alpha)}(\Omega)$ *is open in* $\mathbb{S}_p^{\rho(\alpha)}(\Omega)$.

Proof: Observe that π_p induces naturally a continuous linear map $\hat{\pi} : \mathbb{S}_p^{\rho(\alpha)}(\Omega) \to E(\Omega)$ so that $\mathcal{S}_p^{\rho(\alpha)}(\Omega) = \hat{\pi}^{-1}(\mathcal{E}(\Omega))$. Hence the assertion follows from Theorem 2.1. ▮

Let $\overline{X} := (X_0, X_1)$ be a pair of Banach spaces with $X_1 \overset{d}{\hookrightarrow} X_0$. Then we denote by $\mathcal{H}(\overline{X})$ the set of all $A \in \mathcal{L}(X_1, X_0)$ with $A \in \mathcal{H}(X_0)$, where now A is considered as a linear operator in X_0, of course. A subset $\mathfrak{A}$ of $\mathcal{H}(\overline{X})$ is said to be *regularly bounded* if

(i) $\mathfrak{A}$ is bounded in $\mathcal{L}(X_1, X_0)$;

(ii) There exist constants M and ω so that $[\mathrm{Re}\,\lambda \geq \omega] \subset \rho(-A)$ and

$$\|(\lambda + A)^{-1}\|_{\mathcal{L}(X_0)} \leq M(1 + |\lambda|)^{-1}, \quad \mathrm{Re}\,\lambda \geq \omega, \quad A \in \mathfrak{A} \ ;$$

(iii) $\{(\omega + A)^{-1}\,;\, A \in \mathfrak{A}\}$ is bounded in $\mathcal{L}(X_0, X_1)$.

It is now easy to prove the following basic

Theorem 6.3. *Suppose that $1 \leq 2\alpha < (1+1/p) \wedge (2-n/\hat{p})$. The map*

$$A_{\alpha-1} : \mathcal{S}_p^{\rho(\alpha)}(\Omega) \to \mathcal{H}(E_{\alpha-1}, E_\alpha) \quad , \ \sigma \mapsto A_{\alpha-1}(\sigma) \ , \tag{6.4}$$

is well defined and analytic. If B is a bounded subset of $\mathcal{S}_p(\Omega)$, such that $\pi_p(B)$ is bounded away from $\partial\mathcal{E}(\Omega)$, then there exists a neighbourhood U of B in $\mathcal{S}_p(\Omega)$ so that $A_{\alpha-1}(U)$ is regularly bounded.

Proof: The last part of the assertion is an easy consequence of Theorems 5.2 and 5.6. Observe that Theorem 6.1 and Lemma 6.2 imply that the map (6.4) is the restriction of a continuous linear map to an open subset of the Banach space $\mathbb{S}_p^{\rho(\alpha)}(\Omega)$. Since it follows from [15, Lemma 4.1] that $\mathcal{H}(E_{\alpha-1}, E_\alpha)$ is open in $\mathcal{L}(E_\alpha, E_{\alpha-1})$, this map is analytic. ∎

Remark 6.4. The only place where we made use of the assumption that Ω be of class C^∞ is in the proof of Proposition 5.5 where we referred to [36, 18, 19]. In the latter papers the authors proved their results in the C^∞-category although weaker regularity assumptions would suffice.

Part Three: Quasilinear problems.

7. General existence results. We fix $p \in (1, \infty)$ and introduce assumption **(A1)**:

$1 \leq s < (1+1/p) \wedge (2-n/\hat{p})$;
V is a nonempty open subset of $H^s_{p,\mathcal{B}}$;
Λ is a metric space.

If $-2+1/p < r \leq s$ and $r \notin \mathbb{Z} + 1/p$, we put

$$V_r := V, \text{ equipped with the topology induced by } H^r_{p,\mathcal{B}} \ .$$

Moreover we use the notations and conventions of [15], in particular the ones explained in the beginning of Sections 6 and 11 and of [15].

We denote by T a positive number and introduce assumption **(A2)**:

$\hat{\sigma}(\cdot) \in C([0,T] \times V \times \Lambda, \mathbb{S}_p(\Omega))$ and $\hat{\sigma}(\cdot)$ is bounded on bounded sets;
there exist $\mu \in [0,1) \cup \{1-\}$ and numbers r and ρ with

$$\frac{1}{p} \vee \frac{n}{\hat{p}} < r < s < \rho + 1 < (1+1/p) \wedge (2-n/\hat{p})$$

so that
$$\hat{\sigma}(\cdot) \in C^{1-,\mu}(([0,T] \times V_r) \times \Lambda \ \ , \ \mathcal{S}_p^{\hat{\rho}}(\Omega)) \ .$$
for some $\hat{\rho} \in (\rho, 1)$.

By abusing notation we put

$$(\mathcal{A}(t,v,\lambda), \mathcal{B}(t,v,\lambda)) := (\mathcal{A}(\hat{\sigma}(t,v,\lambda)), \mathcal{B}(\hat{\sigma}(t,v,\lambda))) \ ,$$

or, more explicitly,

$$\mathcal{A}(t,v,\lambda)u = -\partial_j(a_{jk}(\cdot,t,v,\lambda)\partial_k u) + a_j(\cdot,t,v,\lambda)\partial_j u + a_0(\cdot,t,v,\lambda)u \ ,$$

$$\mathcal{B}(t,v,\lambda)u = \delta(a_{jk}(\cdot,t,v,\lambda)\nu^j\partial_k u + b_0(\cdot,t,v,\lambda)\gamma u) + (1-\delta)\gamma u$$

for $(t,v,\lambda) \in [0,T] \times V \times \Lambda$.

The following proposition describes an important special case — to which we refer as 'standard situation' — in which assumptions (A1) and (A2) are satisfied. For simplicity we impose more regularity conditions than actually needed and leave it to the reader to find weaker hypotheses.

Proposition 7.1. (Standard Situation): *Suppose that* $\mathbb{K} = \mathbb{R}$ *and*

- **(i)** $p > n$ *and* $p \geq 2$;
- **(ii)** G *is a nonempty open subset of* $\mathbb{R}^N$ *so that* $\{v \in H^1_{p,\mathcal{B}}\,;\ v(\overline{\Omega}) \subset G\} \neq \emptyset$;
- **(iii)** Λ *is a nonempty open subset of some Banach space* Λ;
- **(iv)** $a_{jk}, a_j \in C^{2-}(\overline{\Omega} \times [0,T] \times G \times \Lambda, \mathcal{L}(\mathbb{R}^N))$,
  $a_0 \in C^{1-}(\overline{\Omega} \times [0,T] \times G \times \Lambda, \mathcal{L}(\mathbb{R}^N))$,
  $b_0 \in C^{2-}(\partial\Omega \times [0,T] \times G \times \Lambda, \mathcal{L}(\mathbb{R}^N))$,
- **(v)** $((a_{jk}(\cdot,t,y,\lambda)), (a_{jk}(\cdot,t,y,\lambda)\nu^j)_{1\leq k\leq n}, 1) \in \mathcal{E}(\Omega)$ *for* $(t,y,\lambda) \in [0,T] \times G \times \Lambda$;
- **(vi)** $n/p < r < s < (1+1/p) \wedge (2-n/p)$, $0 \leq s-1 < \rho < (r-n/p) \wedge 1/p$.

Then assumptions (A1) *and* (A2) *are satisfied with*

$$V := \{v \in H^s_{p,\mathcal{B}}\,;\ v(\overline{\Omega}) \subset G\}\ , \tag{7.1}$$

$\mu := 1-$, *and* $\hat{\sigma}(\cdot)$ *being the 'substitution map', that is,*

$$\hat{\sigma}(t,v,\lambda)(x) := ((a_{jk}), (a_1,\dots,a_n), a_0, b_0)(x,t,v(x),\lambda)$$

for $x \in \overline{\Omega}$ *and* $(t,v,\lambda) \in [0,T] \times V \times \Lambda$.

Proof: Observe that (i) implies $1/p = 1/\hat{p}$. Hence it follows from (vi) that r, s and ρ satisfy the inequalities specified in (A1) and (A2). Moreover

$$H^s_{p,\mathcal{B}} \hookrightarrow H^r_{p,\mathcal{B}} \hookrightarrow C^{r-n/p}_{\mathcal{B}} := \{u \in C^{r-n/p}(\overline{\Omega}, \mathbb{R}^N)\,;\ (1-\delta)\gamma u = 0\}\ .$$

Now the assertion is a consequence of the mean value theorem (cf. the proofs of [7, Propositions 15.4 and 15.6]). ∎

Finally we introduce assumption **(A3)**:

(i) There exists a number β_0 with $2\beta_0 \in (\rho-1, 0]$ so that

$$F \in C^{1-,\mu}(([0,T] \times V_r) \times \Lambda, H^{2\beta_0}_{p,\mathcal{B}})\ .$$

(ii) Given $\lambda \in \Lambda$, there are numbers $\alpha_0 := (\rho+1)/2 < \alpha_1 < \alpha_2 < \dots < \alpha_m := 1$ and $\beta_0 \leq \beta_1 \leq \dots \leq \beta_m < (1+1/p)/2$ with $2\alpha_{j+1}, 2\beta_j \notin \mathbb{Z} + 1/p$, satisfying

$$2(\alpha_{j+1} - \alpha_j) < \rho + 1 - r \quad \text{and} \quad 0 < \alpha_{j+1} - \beta_j < 1$$

for $j = 0, 1, \dots, m-1$, such that

$$F(\cdot,\cdot,\lambda) \in C([0,T] \times (V \cap H^{2\alpha_j}_p), H^{2\beta_j}_{p,\mathcal{B}})\ \ ,\ j = 0,1,\dots,m\ , \tag{7.2}$$

where $(V \cap H^{2\alpha_j}_p)$ is given the topology induced by $H^{2\alpha_j}_p$.

This rather complicated looking assumption — whose second part will be the basis of a bootstrapping argument — can easily be satisfied in the important special case that F is induced by a local function, that is, if F is a substitution operator. This is a consequence of the following proposition, for which we impose again more regularity conditions than really needed.

Proposition 7.2. (Standard Situation): *Suppose that* $\mathbb{K} = \mathbb{R}$ *and*

(i) $p > n$ *and* $p \geq 2$;

(ii) G *is a nonempty open subset of* $\mathbb{R}^N$ *such that* $V := \{u \in H^1_{p,\mathcal{B}}\,;\, u(\overline{\Omega}) \subset G\} \neq \emptyset$;

(iii) Λ *is a nonempty open subset of some Banach space* $\mathbf{\Lambda} := (\mathbf{\Lambda}, |\cdot|)$;

(iv) $n/p < r < s < (1 + 1/p) \wedge (2 - n/p)$, $0 \leq s - 1 < \rho < (r - n/p) \wedge 1/p$.

Moreover, let one of conditions (a) and (b) below be satisfied.

(a) $s = 1$, $f \in C^{2-}(\overline{\Omega} \times [0,T] \times G \times \Lambda, \mathbb{R}^N)$, *and* $F(t, v, \lambda) := f(\cdot, t, v(\cdot), \lambda)$, $(t, v, \lambda) \in [0,T] \times V \times \Lambda$.

(b) $r = 1$, $f \in C^{2-}(\overline{\Omega} \times [0,T] \times G \times \mathbb{R}^{nN} \times \Lambda, \mathbb{R}^N)$, *there exists an increasing function* $c : \mathbb{R}^+ \to \mathbb{R}^+$ *and a constant* κ *so that* $1 \leq \kappa < 1 + p/n$ *and*

$$(|f| + |\partial_2 f| + |\partial_3 f| + (1 + |\eta|)|\partial_4 f| + |\partial_5 f|)(x, t, \xi, \eta, \lambda) \leq c(|\xi| + |\lambda|)(1 + |\eta|^\kappa)$$

for all $(x, t, \xi, \eta, \lambda) \in \overline{\Omega} \times [0,T] \times G \times \mathbb{R}^{nN} \times \Lambda$, $\rho < 1 - (\kappa - 1)n/p$, *and*

$$F(t, v, \lambda) := f(\cdot, t, v(\cdot), \partial v(\cdot), \lambda), \quad (t, v, \lambda) \in [0,T] \times V \times \Lambda .$$

Then F satisfies assumption $(A3)$ with $\mu = 1-$.

Proof: Assume that (a) is satisfied. Observe that

$$H^\sigma_p \hookrightarrow H^r_p \hookrightarrow C^{r-n/p} \hookrightarrow H^\varepsilon_p \hookrightarrow L_p \hookrightarrow H^\tau_p \tag{7.3}$$

for $\sigma > r$, $r - n/p > \varepsilon > 0 > \tau > -1 + 1/p$. Similarly as in the proof of [7, Propositions 15.4 and 15.6] one sees that

$$F \in C^{1-}([0,T] \times (V \cap C^\alpha) \times \Lambda, C^\alpha) \tag{7.4}$$

for $0 \leq \alpha < 1$, where $V \cap C^\alpha$ is given the C^α-topology. Using these facts, the assertion follows easily in this case.

Suppose now that (b) is fulfilled. It follows from Sobolev's imbedding theorem and standard duality arguments that

$$H^\sigma_p \hookrightarrow L_\alpha \hookrightarrow L_\beta \hookrightarrow H^\tau_{p,\mathcal{B}} \tag{7.5}$$

for $\sigma \in [0,1]$ and $\tau \in (-2 + 1/p, 0] \backslash \{-1 + 1/p\}$, provided

$$\frac{1}{\alpha} := \frac{1}{p} - \frac{\sigma}{n} > 0 \ , \ \frac{1}{\beta} := \frac{1}{p} - \frac{\tau}{n} > 0$$

(cf. [14, Lemma 14.2]). Observe that

$$\frac{\alpha}{\beta} = 1 + \frac{p(\sigma - \tau)}{n - p\sigma} \tag{7.6}$$

is an increasing function of $\sigma \in [0, n/p)$.

Put $\hat{f}(t, v, w, \lambda) := f(\cdot, t, v(\cdot), w(\cdot), \lambda)$ for $(v, w) \in (V \cap C) \times (L_1)^n$ and $(t, \lambda) \in [0, T] \times \Lambda$, where $V \cap C$ is given the C-topology. Then it is not difficult to see that the growth restrictions for f imply

$$\hat{f} \in C^{1-}([0, T] \times (V \cap C) \times (L_\alpha)^n \times \Lambda, L_\beta) \ , \tag{7.7}$$

provided $\kappa \leq \alpha/\beta$ (cf. the proof of [14, Lemma 14.1]). Hence it follows from (7.5), the commutativity of the diagramm

$$\begin{array}{ccccc} V \cap H_p^{\sigma+1} & \overset{(id, \partial)}{\longrightarrow} & (V \cap H_p^{\sigma+1}) \times (H_p^\sigma)^n & \hookrightarrow & (V \cap C) \times (L_\alpha)^n \\ F \searrow & & & & \swarrow \hat{f} \\ & H_{p,\mathcal{B}}^\tau & \longleftarrow & L_\beta & \end{array}$$

(where we suppressed the factor $[0, T] \times \Lambda$ everywhere in the first row), and (7.7) that

$$F \in C^{1-}([0, T] \times (V \cap H_p^{\sigma+1}) \times \Lambda, H_{p,\mathcal{B}}^\tau) \ , \tag{7.8}$$

provided σ and τ satisfy the restrictions specified above and $\kappa \leq \alpha/\beta$.

Choose $2\beta_0 \in (\rho - 1, 0]$ so that $1 - 2\beta_0 p/n \geq \kappa$, which is possible since $1 \leq 1 - 2\beta_0 p/n < 1 + (1 - \rho)p/n$ for $2\beta_0 \in (\rho - 1, 0]$ and since $1 + (1 - \rho)p/n > \kappa$ by our assumption on ρ. Then it follows from (7.6) and (7.8) (with $\sigma := 0$ and $\tau := 2\beta_0$) that

$$F \in C^{1-}([0, T] \times V_r \times \Lambda, H_{p,\mathcal{B}}^{2\beta_0}) \ .$$

Hence assumption (A3 i) is satisfied with $\mu = 1-$.

Put $2\alpha_0 := 1 + \rho$. It is easily seen that we can find finite sequences $\alpha_0 < \alpha_1 < \ldots < \alpha_m = 1$ and $\beta_0 \leq \beta_1 \leq \ldots \leq \beta_m < (1 + 1/p)/2$ so that $2\alpha_j$, $2\beta_j \notin \mathbb{Z} + 1/p, 2(\alpha_{j+1} - \alpha_j) < \rho + 1 - r = \rho$, $0 < \alpha_{j+1} - \beta_0 < 1$, and — as long as $2\alpha_j < 1 + n/p$ —

$$1 + \frac{(2\alpha_j - 1 - 2\beta_j)p}{n - (2\alpha_j - 1)p} \geq \kappa \ .$$

Then it follows from (7.6), letting $\sigma := 2\alpha_j - 1$ and $\tau := 2\beta_j$ as long $2\alpha_j < 1 + n/p$, and from (7.3) and similar arguments as those leading to (7.4), if $2\alpha_j > 1 + n/p$, that assumption (A3 ii) is satisfied. ∎

Observe that $\kappa \geq 2$ so that we can admit nonlinearities 'growing quadratically in the gradient' for every choice of $p > n$. However we are free to choose p as large as we want to. Thus, in the standard situation we can admit nonlinearities with arbitrary polynomial growth in the gradient (and no growth restriction in the function itself), provided we put $r = 1$ and, consequently, $s > 1$.

We assume now that (A1) – (A3) are satisfied. Then we consider the *Cauchy problem for the parameter dependent quasilinear reaction-diffusion system*

$$\begin{aligned} \partial_t u + \mathcal{A}(t,u,\lambda)u = F(t,u,\lambda) \quad & \text{in } \Omega \times (\tau, T] \ , \\ \mathcal{B}(t,u,\lambda)u = 0 \quad & \text{on } \partial\Omega \times (\tau, T] \ , \\ u(0) = u_0 \quad & \text{on } \Omega \ , \end{aligned} \qquad (QRDS)_{(\tau,u_0,\lambda)}$$

where $\tau \in [0,T)$, $u_0 \in V$, and $\lambda \in \Lambda$, are given. By an *(L_p-) solution u of $(QRDS)_{(\tau,u_0,\lambda)}$ on J* we mean a function

$$u \in C(J,V) \cap C(\dot{J}, H_p^2) \cap C^1(\dot{J}, L_p) \ , \tag{7.9}$$

where J is a subinterval of $[\tau, T]$ containing τ, so that $\dot{J} := J\backslash\{\tau\} \neq \emptyset$, which satisfies $(QRDS)_{(\tau,u_0,\lambda)}$ pointwise (in $t \in J$). A solution u is *maximal* if there does not exist a solution which is a proper extension of u. Observe that u is a solution of $(QRDS)_{(\tau,u_0,\lambda)}$ on J if and only if u is a solution on J of the abstract *quasilinear Cauchy problem (in L_p)*

$$\dot{u} + A(t,u,\lambda)u = F(t,u,\lambda) \ \ , \ \tau < t \leq T \ , \ u(\tau) = u \ , \qquad (QCP)_{(\tau,u_0,\lambda)}$$

that is, u satisfies (7.9), $u(t) \in D(A(t,u(t),\lambda)) = H^2_{p,\mathcal{B}(t,u(t),\lambda)}$ for $t \in \dot{J}$, and u satisfies $(QCP)_{(\tau,u_0,\lambda)}$ pointwise on J.

In the following we shall say that (A1) - (A3) *are true for every $T > 0$* if we can replace the interval $[0,T]$ by $\mathbb{R}^+$. In this case we drop the restrictions $t \leq T$ in $(QRDS)_{(\tau,u_0,\lambda)}$ and $(QCP)_{(\tau,u_0,\lambda)}$ everywhere, of course. In particular, (A1) - (A3) are true for every $T > 0$ if $\hat{\sigma}(\cdot)$ and F are independent of t. In this case we obtain the *autonomous reaction-diffusion system*

$$\begin{aligned} \partial_t u + \mathcal{A}(u,\lambda)u = F(u,\lambda) \quad & \text{in } \Omega \times (0,\infty) \ , \\ \mathcal{B}(u,\lambda)u = 0 \quad & \text{on } \partial\Omega \times (0,\infty) \ , \\ u(0) = u_0 \quad & \text{on } \Omega \ , \end{aligned} \qquad (RDS)_{(u_0,\lambda)}$$

which is equivalent to the *autonomous quasilinear Cauchy problem*

$$\dot{u} + A(u,\lambda)u = F(u,\lambda) \ \ , \ 0 < t < \infty \ , \ u(0) = u_0 \qquad (CP)_{(u_0,\lambda)}$$

(in L_p, of course).

After these preparations we can formulate the basic existence, uniqueness, and continuity theorem whose proof will be given in Section 8 below.

Theorem 7.3. (i) *Problem $(QRDS)_{(\tau,u_0,\lambda)}$ possesses for each $(\tau,u_0,\lambda) \in [0,T) \times V \times \Lambda$ a unique maximal solution $u(\cdot,\tau,u_0,\lambda)$. The maximal interval of existence, $J := J(\tau,u_0,\lambda)$, is open in $[\tau,T]$. The set*

$$\mathcal{D}(\tau) := \{(t,v,\lambda) \in [\tau,T] \times V \times \Lambda \, ; \, t \in J(\tau,v,\lambda)\}$$

is open in $[\tau,T] \times V \times \Lambda$ and

$$u(\cdot,\tau,\cdot,\cdot) \in C^{0,1-,\mu}(\mathcal{D}(\tau))$$

for $\tau \in [0,T)$.

(ii) *If $u_0 \in H^2_{p,\mathcal{B}(\tau,u_0,\lambda)}$, then*

$$u(\cdot,\tau,u_0,\lambda) \in C(J,H^2_p) \cap C^1(J,L_p) \ . \tag{7.10}$$

(iii) *If $u(J,\tau,u_0,\lambda)$ is bounded in H^s_p and bounded away from ∂V, then $u(\cdot,\tau,u_0,\lambda)$ is a global solution, that is, $J = [\tau,T]$.*

(iv) *Suppose that (A1) - (A3) are true for every $T > 0$. Then $u(\cdot,\tau,u_0,\lambda)$ is a global solution, that is, $J = [\tau,\infty)$, provided $u(\cdot,\tau,u_0,\lambda)$ is, on each bounded interval, bounded in H^s_p and bounded away from ∂V .*

(v) *Suppose also that, for each fixed $\lambda \in \Lambda$, and for each bounded subset B of V, which is bounded away from ∂V,*

$$\hat{\sigma}(\mathbb{R}^+ \times B \times \{\lambda\}) \text{ is bounded in } \mathbb{S}_p(\Omega) \ , \tag{7.11}$$

$$\hat{\sigma}(\cdot,\cdot,\lambda) \in BUC^{1-}(\mathbb{R}^+ \times B_r, \mathcal{S}^{\hat{\rho}}_p(\Omega)) \ , \tag{7.12}$$

and

$$F(\mathbb{R}^+ \times B \times \{\lambda\}) \text{ is bounded in } H^{2\beta_0}_{p,\mathcal{B}} \ . \tag{7.13}$$

If $u(\cdot,\tau,u_0,\lambda)$ is uniformly bounded in H^s_p and uniformly bounded away from ∂V, then $u(\cdot,\tau,u_0,\lambda)$ is uniformly bounded in $H^{\rho+1}_p$ on $[\tau',\infty)$ for every $\tau' > \tau$. If, in addition,

$$F(\mathbb{R}^+ \times B_j \times \{\lambda\}) \text{ is bounded in } H^{2\beta_j}_{p,\mathcal{B}} \ \ , \ j = 0,1,\dots,m \ , \tag{7.14}$$

for every bounded subset B_j of $(V \cap H^{2\alpha_j}_p)$, which is bounded away from ∂V, then $u(\cdot,\tau,u_0,\lambda)$ is uniformly bounded in H^2_p on $[\tau',\infty)$ for every $\tau' > \tau$.

In the autonomous case we put

$$\varphi(t,v,\lambda) := u(t,0,v,\lambda) \ \ , \ t^+(v,\lambda) := \sup J(0,v,\lambda) \ . \tag{7.15}$$

Moreover we put

$$\mathcal{D} := \mathcal{D}(0) \ \text{ and } \gamma^+(v,\lambda) := \varphi([0,t^+(v,\lambda)),v,\lambda) \ .$$

Observe that $t^+(v,\lambda) = \infty$ if the 'orbit' $\gamma^+(v,\lambda)$ is bounded in H^s_p and bounded away from ∂V, thanks to Theorem 7.3 (iv). In the following we shall say that 'the orbit $\gamma^+(v,\lambda)$ is bounded in H^σ_p for $t > 0$', where $\sigma \in (s,2]$, provided $\varphi\big([\tau,t^+(v,\lambda)),$ $v,\lambda\big)$ is bounded in H^σ_p for every $\tau \in (0,t^+(v,\lambda))$. Observe that $\varphi((0,t^+(v,\lambda)),v,\lambda)$ $\subset H^2_p$ for every $(v,\lambda) \in V \times \Lambda$, thanks to Theorem 7.3.

Corollary 7.4. *The parameter dependent autonomous reaction-diffusion system $(RDS)_{(u_0,\lambda)}$ generates for each $\lambda \in \Lambda$ a semiflow $\varphi(\cdot,\cdot,\lambda)$ on V, given by (7.12), so that*

$$\varphi \in C^{0,1-,\mu}(\mathcal{D},V) \ .$$

Bounded orbits, which are bounded away from ∂V, are bounded in $H^{1+\rho}_p$ for $t > 0$ and relatively compact in V. If

$$F(B_j \times \{\lambda\}) \text{ is bounded in } H^{2\beta_j}_{p,\mathcal{B}} \ \ , \ j = 0,1,\dots,m \ , \tag{7.16}$$

for every $\lambda \in \Lambda$ and every bounded subset B_j of $(V \cap H_p^{2\alpha_j})$, which is bounded away from ∂V, then bounded orbits, which are bounded away from ∂V, are bounded in H_p^2 for $t > 0$. If $\varphi([0, T \wedge t^+(v,\lambda)), v, \lambda)$ is bounded and bounded away from ∂V for every $T > 0$, then $t^+(v,\lambda) = \infty$.

Observe that the boundedness of $\varphi([0, T \wedge t^+(v,\lambda)), v, \lambda)$ for every $T > 0$ does not imply the boundedness of the orbit $\gamma^+(v,\lambda)$ if $t^+(v,\lambda) = \infty$.

Proof: Observe that

$$H_{p,\mathcal{B}}^{\rho+1} \subset\subset\!\!\hookrightarrow H_{p,\mathcal{B}}^{s} \subset\subset\!\!\hookrightarrow H_{p,\mathcal{B}}^{r} , \tag{7.17}$$

where $\subset\subset\!\!\hookrightarrow$ denotes 'compact injection' (cf. (8.4) below). Hence, if B is a bounded subset of V, which is bounded away from ∂V, it follows from the second part of (7.17) that B is relatively compact in V_r. Since Lipschitz continuous maps are uniformly Lipschitz continuous on relatively compact sets, it follows now from the last part of assumption (A2) and the fact that $\hat{\sigma}$ is independent of $t \in \mathbb{R}^+$, that (7.12) is satisfied. Assumption (A3 i), the relative compactness of B in V_r, and the independence of $\hat{\sigma}$ of $t \in \mathbb{R}^+$ imply (7.13), whereas (7.11) is an immediate consequence of the first part of assumption (A2). Now the assertion is implied by Theorem 7.4 and the first part of (7.17). ∎

8. Proof of the existence theorem. Fix $s_0 \in (s-2, \rho-1)$ arbitrarily, put $\alpha := (s - s_0)/2$ and $\varepsilon := (\rho - 1 - s_0)/2$, as well as

$$\mathbb{E}_{\alpha-1} := H_{p,\mathcal{B}}^{s-2} \quad , \quad \mathbb{E}_{1+\varepsilon} := H_{p,\mathcal{B}}^{\rho+1} .$$

Moreover, define $\mathbb{E}_\xi$ for $\alpha - 1 \le \xi \le 1 + \varepsilon$ by

$$\mathbb{E}_\xi := [\mathbb{E}_{\alpha-1}, \mathbb{E}_{1+\varepsilon}]_{(\xi-\alpha+1)/(2+\varepsilon-\alpha)} .$$

Then the reiteration theorem for the complex interpolation functor implies

$$\mathbb{E}_\eta = [\mathbb{E}_\xi, \mathbb{E}_\zeta]_{(\eta-\xi)/(\zeta-\xi)} , \quad \alpha - 1 \le \xi \le \eta \le \zeta \le 1 + \varepsilon . \tag{8.1}$$

Moreover we deduce from (5.4) and Proposition (5.5) that

$$\mathbb{E}_\xi \doteq H_{p,\mathcal{B}}^{s_0+2\xi} \quad , \quad \alpha - 1 \le \xi \le 1 + \varepsilon \ , \ s_0 + 2\xi \notin \mathbb{Z} + 1/p . \tag{8.2}$$

We define real numbers β and γ by

$$2\beta := r - s_0 \quad , \quad 2\gamma := 2\beta_0 - s_0 ,$$

and observe that

$$0 < \varepsilon < \gamma < \beta < \alpha < 1 . \tag{8.3}$$

Next we set

$$\mathbb{A}_\xi := \mathbb{A}_\xi(\cdot,\cdot,\cdot) := A_{\xi+s_0/2}(\cdot,\cdot,\cdot) \quad , \quad \alpha - 1 \le \xi \le \varepsilon ,$$

and $\mathbb{A} := \mathbb{A}_0$. Finally we put $V_{(\xi)} := V \cap \mathbb{E}_\xi$, endowed with the topology induced by $\mathbb{E}_\xi$. Observe that $V_{(\alpha)} = V_s = V$ and $V_{(\beta)} = V_r$.

Lemma 8.1. (i) $\mathbb{A}_\xi \in C^{1-,\mu}(([0,T] \times V_{(\beta)}) \times \Lambda, \mathcal{L}(\mathbb{E}_{1+\xi}, \mathbb{E}_\xi))$ *for* $\alpha - 1 \leq \xi \leq \varepsilon$.

(ii) *The map*

$$[(v,\lambda) \mapsto \mathbb{A}_\xi(\cdot, v, \lambda)] : V_{(\alpha)} \times \Lambda \to C_T^{1-}(\mathcal{H}(\mathbb{E}_\xi, \mathbb{E}_{1+\xi}))$$

is well defined and locally regularly bounded for each $\xi \in [\alpha - 1, \varepsilon]$.

(iii) *The map*

$$[v \mapsto \mathbb{A}_\xi(\cdot, v, \lambda)] : V_{(\alpha)} \to C_T^{1-}(\mathcal{H}(\mathbb{E}_\xi, \mathbb{E}_{1+\xi}))$$

is, for each $\xi \in [\alpha - 1, \varepsilon]$ *and* $\lambda \in \Lambda$, *regularly bounded on bounded subsets which are bounded away from* $\partial V_{(\alpha)}$.

Proof: (i) is an easy consequence of assumption (A2) and Theorem 6.3.

(ii) and (iii): It follows from the Rellich-Kondrachev theorem (e.g. [2, Theorem 6.2]) and well known facts from interpolation theory (e.g. [16, Corollary 3.8.2 and Theorems 4.7.1 and 3.4.1]) that

$$\mathbb{E}_\eta \subset\subset\hookrightarrow \mathbb{E}_\zeta \quad , \quad \alpha - 1 \leq \zeta < \eta \leq 1 + \varepsilon \ . \tag{8.4}$$

Let B be a bounded subset of $V_{(\alpha)}$ which is bounded away from $\partial V_{(\alpha)}$. Then we deduce from (8.4) that $\overline{B}_\beta$, the closure of B in $\mathbb{E}_\beta$, is a compact subset of $V_{(\beta)}$. Hence, given any $\lambda \in \Lambda$, the second part of assumption (A2) implies the compactness of $\hat{B} := \hat{\sigma}([0,T] \times \overline{B}_\beta \times \{\lambda\})$ in $\mathcal{S}_p^{\hat{\rho}}(\Omega)$. Observe that

$$\pi_p(\sigma) = ((a_{jk}(t,v,\lambda)) \quad , \quad (a_{jk}(t,v,\lambda)\nu^j)_{1 \leq k \leq n}, 1)$$

for $\sigma := \hat{\sigma}(t,v,\lambda) \in \hat{B}$. Hence we see that $\pi_p(\hat{B})$ is relatively compact in $\mathcal{E}(\Omega)$. Since the first part of assumption (A2) implies the boundedness of $\hat{B}$ in $\mathcal{S}_p(\Omega)$, we deduce from Theorem 6.3 the existence of a neighbourhood $\hat{U}$ of $\hat{B}$ in $\mathcal{S}_p(\Omega)$ so that $\mathbb{A}_\xi(\hat{U})$ is regularly bounded. Hence the first part of assumption (A2) implies the existence of a neighbourhood U of λ in Λ so that $\mathbb{A}_\xi([0,T] \times B \times U)$ is regularly bounded. By using the compactness of $[0,T] \times \overline{B}_\beta$ and the second part of assumption (A2) it is easily seen that we can find a constant L — by making U smaller if necessary — so that

$$\|\mathbb{A}_\xi(t,v,\lambda) - \mathbb{A}_\xi(t',v,\lambda)\|_{\mathcal{L}(\mathbb{E}_{1+\xi},\mathbb{E}_\xi)} \leq L|t - t'|$$

for $t, t' \in [0,T]$ and $v, \lambda \in B \times U$. This proves the assertions. ∎

Before proving the next lemma we point out that the spaces $E_{1+\xi}(t,y,\lambda)$, $\varepsilon < \xi < 1$, of [15, Section 9] have nowhere been used in [15]. For the construction of $E_{1+\xi}(t,y,\lambda)$, $0 < \xi \leq \varepsilon$, the spaces $E_2(t,y,\lambda)$ were needed. However it is completely irrelevant how the spaces $E_{1+\xi}(t,y,\lambda)$, $0 < \xi \leq \varepsilon$, are obtained, as long as they satisfy condition (Q3). Using this observation we can prove

Lemma 8.2. *Assumptions* (Q1) - (Q4) *and* (B) *of* [15] *are satisfied (where* $\overline{E}$ *corresponds now to* $(\mathbb{E}_0, \mathbb{E}_1)$, *and* A *to* $\mathbb{A}$, *respectively).*

Proof: The validity of (Q1) - (Q4) follows from assumptions (A1) and (A2), from (8.1) and (8.3), the reflexivity of $\mathbb{E}_\xi$, $\alpha - 1 \leq \xi \leq 1 + \varepsilon$, Lemma 8.1 (i) and (ii), and assumptions (A3 (i)). The validity of (B) is an easy consequence of (8.4), the continuity of F, and Lemma 8.1 (iii). ∎

Proof of Theorem 7.3: We consider the abstract quasilinear Cauchy problem

$$\dot{u} + \mathbb{A}(t,u,\lambda)u = F(t,u,\lambda) \quad , \ \tau < t \leq T \ , \ u(\tau) = u_0 \ . \tag{8.5}$$

Thanks to Lemma 8.2 we can apply [15, Theorems 7.1 - 7.3] to this problem. Since every solution of $(QRDS)_{(\tau,u_0,\lambda)}$ is obviously a solution of (8.5), assertion (i) - (iv) follow, provided we show the unique maximal solution $u(\cdot,\tau,u_0,\lambda)$ of (8.5) satisfies (7.9) and, if $u_0 \in H^2_{p,\mathcal{B}(\tau,u_0,\lambda)}$, also (7.10).

We fix (τ,u_0,λ) arbitrarily and omit it from the notation whenever possible.

It is a consequence of [15, Corollary 9.3] that the maximal solution u of (8.5) satisfies

$$u \in C(\dot{J},\mathbb{E}_{1+\varepsilon}) \cap C^1(\dot{J},\mathbb{E}_\varepsilon) \ .$$

Thus, switching back to the notation of Section 5,

$$u \in C(\dot{J},E_{\alpha_0}) \cap C^1(\dot{J},E_{\alpha_0-1}) \ . \tag{8.6}$$

Hence, given $\tau',\tau'' \in \dot{J}$ with $\tau' < \tau''$,

$$\|u(t)-u(t')\|_{\alpha_0} + |t-t'|^{-1}\|u(t)-u(t')\|_{\alpha_0-1} \leq c \quad , \ \tau' \leq t,t' \leq \tau'' \ . \tag{8.7}$$

Recall that $[\cdot,\cdot]_\theta$ is an interpolation functor of exponent θ and that this implies

$$\|x\|_{[X_0,X_1]_\theta} \leq C(\theta)\|x\|_{X_0}^{1-\theta}\|x\|_{X_1}^{\theta} \quad , \ x \in X_0 \cap X_1 \ , \ 0<\theta<1 \ . \tag{8.8}$$

Using Proposition (5.5) and inequality (8.8) we deduce from (8.7) that

$$\|u(t)-u(t')\|_{r/2} \leq c|t-t'|^{\alpha_0-r/2} \quad , \ \tau' \leq t,t' \leq \tau'' \ ,$$

which shows that

$$u \in C^{\alpha_0-r/2}(\dot{J},V_r) \ . \tag{8.9}$$

Suppose that

$$\begin{aligned} &\text{the hypotheses of (iv) and (v) are satisfied and } B := \{u(t)\,;\, t \in J\} \\ &\text{is bounded in } V \text{ and bounded away from } \partial V \ . \end{aligned} \tag{8.10}$$

Then $J = [\tau,\infty)$ by Lemma 8.2 and [15, Theorem 7.3]. Moreover it follows from (7.12) that $\hat{\sigma}(J \times B,\lambda)$ is bounded in $\mathcal{S}_p^{\hat{\rho}}$. Hence we deduce from $C^{\hat{\rho}} \overset{d}{\hookrightarrow} C$ that $\pi_p(\hat{\sigma}(J \times B,\lambda))$ is relatively compact in $\mathcal{E}(\Omega)$. Thus we obtain from (7.11) and Theorem 6.3 that

$$\{A_\eta(t) := A_\eta(t,u(t),\lambda)\,;\, t \in J\} \tag{8.11}$$

is regularly bounded for each η with $s-2 \leq 2\eta \leq \rho-1$. Using this fact, (7.13), the boundedness of B in $H^s_{p,\mathcal{B}}$, and

$$\dot{u}(t) = -A_{(s/2)-1}(t)u(t) + f(t) \quad , \ t \in J \ ,$$

where $f(t) := F(t, u(t), \lambda)$, we find that

$$\sup_{t\in J} \|\dot{u}(t)\|_{(s/2)-1} < \infty \ .$$

Hence there exists a constant c so that

$$\|u(t) - u(t')\|_{s/2} + |t - t'|^{-1}\|u(t) - u(t')\|_{(s/2)-1} \leq c \ , \ t, t' \in J \ .$$

which, thanks to (8.1) and (8.8), implies

$$u \in BUC^{(s-r)/2}(J, H^r_{p,\mathcal{B}}) \ .$$

Now we deduce from (7.12) and Theorem 6.3 that

$$A_\eta(\cdot) \in BUC^{(s-r)/2}(J, \mathcal{L}(H^{2\eta+2}_{p,\mathcal{B}}, H^{2\eta}_{p,\mathcal{B}})) \ , \ s - 2 \leq 2\eta \leq \rho - 1 \ . \tag{8.12}$$

(Observe that the map (6.4) is uniformly Lipschitz continuous, being the restriction of a continuous linear map.) Hence (8.11) and (8.12) show that $\{A_\eta(t)\,;\, t \in J\}$ is, for each η with $s - 2 \leq 2\eta \leq 1 + \rho$, a regularly bounded subset of $C^{(s-r)/2}_T\big(\mathcal{H}(H^{2\eta}_{p,\mathcal{B}}, H^{2\eta+2}_{p,\mathcal{B}})\big)$, uniformly with respect to $T \in \dot{J}$ (cf. [15, Section 4], where we have now replaced $[0, T]$ by $[\tau, T]$, of course.) Hence we can find positive constants M and ω_0 so that

$$\omega_0 + A_{\alpha_0-1}(t) \in \mathcal{H}(H^{2\alpha_0-2}_{p,\mathcal{B}}) \cap \mathcal{G}(H^{2\alpha_0-2}_{p,\mathcal{B}}, M, 0) \ , \ t \in J \ .$$

Observe that this implies

$$\omega_1 + \omega_0 + A_{\alpha_0-1}(t) \in \mathcal{G}(H^{2\alpha_0-2}_{p,\mathcal{B}}, M, -\omega_1) \ , \ t \in J \ ,$$

for every $\omega_1 \in \mathbb{R}$. By choosing ω_1 sufficiently large, setting $\omega := \omega_1 + \omega_0$, and using the uniformity assertions contained in (8.12), we deduce from Theorem A.3 of the Appendix that there exists a constant c so that

$$\|U(t, t')\|_{\mathcal{L}(H^{2\xi}_{p,\mathcal{B}}, H^{2\alpha_0}_{p,\mathcal{B}})} \leq c(t - t')^{\xi-\alpha_0} e^{-(t-t')} \tag{8.13}$$

for $\tau \leq t' < t < \infty$ and $2\xi \in \{s, 2\beta_0\}$, where U is the unique parabolic fundamental solution of

$$\{\omega + A_{\alpha_0-1}(t)\,;\, t \in J\} \ .$$

Thanks to (8.6) we know that u is the solution of

$$\dot{v} + (\omega + A_{\alpha_0-1}(t))v = \omega u(t) + f(t) \ , \ t \in \dot{J} \ , \ v(0) = u_0 \ . \tag{8.14}$$

This implies

$$u(t) = U(t, \tau)u_0 + \int_\tau^t U(t, \tau)(\omega u(\tau) + f(\tau))d\tau \ , \ t \in J \ . \tag{8.15}$$

Hence we deduce from (8.13), the boundedness of u in $H^s_{p,\mathcal{B}}$, from $H^s_{p,\mathcal{B}} \hookrightarrow H^{2\beta_0}_{p,\mathcal{B}}$, and from the boundedness of $f(J)$ in $H^{2\beta_0}_{p,\mathcal{B}}$ the existence of constants c and c_1 so that

$$\|u(t)\|_{\alpha_0} \leq c\Big[(t-\tau)^{(s/2)-\alpha_0}e^{-(t-\tau)} + \int_\tau^t (t-t')^{\beta_0-\alpha_0}e^{-(t-\tau')}\,dt'\Big]$$

$$\leq c_1\big[(t-\tau)^{(s/2)-\alpha_0}e^{-(t-\tau)} + 1\big]$$

for $t \in \dot{J}$, thanks to $\beta_0 > \alpha_0 - 1$. This proves the boundedness of u in $H^{2\alpha_0}_{p,\mathcal{B}}$ on $[\tau', \infty)$ for each $\tau' > \tau$. Consequently, using also (8.12) and (8.14), it follows that

$$u \in BC([\tau', \infty), H^{2\alpha_0}_{p,\mathcal{B}}) \cap BC^1([\tau', \infty), H^{2\alpha_0-2}_{p,\mathcal{B}}) \tag{8.16}$$

for every $\tau' > t$. Hence (8.7) is satisfied for all t and t' in $[\tau', \infty)$, and we deduce from (8.8) that

$$u \in BUC^{\alpha_0 - r/2}([\tau', \infty), V_r) \quad , \ \tau' > \tau \ . \tag{8.17}$$

This proves the first part of the assertion of (v).

We return now to the general case. It follows from (8.9) and Lemma (8.1) (i) that

$$A_{\alpha_0-1}(\cdot) \in C^{\alpha_0-r/2}(\dot{J}, \mathcal{L}(H^{2\alpha_0}_{p,\mathcal{B}}, H^{2\alpha_0-2}_{p,\mathcal{B}})) \ . \tag{8.18}$$

If condition (8.10) is also satisfied then (8.17), (7.11), (7.12), and Theorem 6.3 imply

$$A_{\alpha_0-1}(\cdot) \in BUC^{\alpha_0-r/2}([\tau', \infty), \mathcal{L}(H^{2\alpha_0}_{p,\mathcal{B}}, H^{2\alpha_0-2}_{p,\mathcal{B}})) \quad , \ \tau' > \tau \ . \tag{8.19}$$

Since $\rho+1-r = 2\alpha_0 - r > 2(\alpha_1 - \alpha_0)$ and $\beta_0 > \alpha_1 - 1$, we deduce from [13, Theorem 8.2] (by identifying there α and $\alpha_1 - 1, \beta$ and $\alpha_0 - 1$, and γ and β_0, respectively) that u is a solution on J of

$$\dot{u} + A_{\alpha_1-1}(t)u = f(t) \quad , \ t \in \dot{J} \ . \tag{8.20}$$

Thus

$$u \in C^1(\dot{J}, H^{2\alpha_1-2}_{p,\mathcal{B}}) \ . \tag{8.21}$$

Suppose first that $2\alpha_1 < 1 + 1/p$. Then we know from [13, Theorem 8.2] also that

$$u \in C(\dot{J}, H^{2\alpha_1}_{p,\mathcal{B}}) \ . \tag{8.22}$$

Hence, given any $\tau', \tau'' \in \dot{J}$ with $\tau' < \tau''$, we deduce from (8.21) and (8.22) the existence of a constant c so that

$$\|u(t) - u(t')\|_{\alpha_1} + |t-t'|^{-1}\|u(t) - u(t')\|_{\alpha_1-1} \leq c \quad , \ \tau' \leq t, t' \leq \tau'' \ , \tag{8.23}$$

which is the same estimate as (8.7), except that α_0 has been replaced by α_1. Hence it follows that

$$u \in C^{\alpha_1 - r/2}(\dot{J}, V_r) \ . \tag{8.24}$$

Suppose now that $2\alpha_1 > 1 + 1/p$. Given any $\tau'' \in \dot{J}$, it follows from (8.4) and $u \in C(J, V)$, similarly as in the proof of Lemma 8.1, that

$$B_1 := \{\hat{\sigma}(t, u(t), \lambda) \, ; \, \tau \le t \le \tau''\}$$

is a bounded subset of $\mathcal{S}_p(\Omega)$ such that $\tau_p(B_1)$ is relatively compact in $\mathcal{E}(\Omega)$. Hence Theorem 5.2 and Corollary 5.7 imply the existence of constants κ and ω so that

$$\|u(t)\|_{2\alpha_1,p} \le \kappa \|(\omega + A_{\alpha_1 - 1}(t))u(t)\|_{\alpha_1 - 1} \quad , \quad \tau \le t \le \tau'' \ . \tag{8.25}$$

Thus, given any $\tau' \in (\tau, \tau'')$, it follows from (8.20) and (8.21) that there is a constant c so that

$$\|u(t)\|_{2\alpha_1,p} \le \kappa(\|f(t)\|_{\alpha_1-1} + \omega\|u(t)\|_{\alpha_1-1} + \|\dot{u}(t)\|_{\alpha_1-1}) \le c \tag{8.26}$$

for $\tau' \le t \le \tau''$. Since $H^{2\alpha_1-2}_{p,\mathcal{B}} = H^{2\alpha_1-2}_p$, due to $-1 + 1/p < 2\alpha_1 - 2 \le 0$, we see that (8.21) and (8.26) imply the existence of a constant c so that

$$\|u(t) - u(t')\|_{2\alpha_1,p} + |t - t'|^{-1}\|u(t) - u(t')\|_{2\alpha_1-2,p} \le c \quad , \quad \tau' \le t, t' \le \tau'' \ . \tag{8.27}$$

Now we can use the fact that

$$H^\eta_p \doteq [H^\xi_p, H^\zeta_p]_{(\eta-\xi)/(\zeta-\xi)} \quad , \quad -1 + 1/p < \xi < \eta < \zeta \le 2 \ , \tag{8.28}$$

together with the closedness of $H^r_{p,\mathcal{B}}$ in H^r_p, to deduce the validity of (8.24) also in this case. Observe that the estimate (8.27) remains valid (with a different constant c, of course) if we replace there α_1 by any number $\gamma > \alpha_1$ so that $2\alpha_0 - r > 2(\gamma - \alpha_0)$ and $\beta_0 > \gamma - 1$. Hence we deduce from this new estimate and (8.28) that u is continuous from $\dot{J}$ into $H^{2\alpha_1}_p$. Hence

$$u \in C(\dot{J}, H^{2\alpha_1}_p) \cap C^{\alpha_1 - r/2}(\dot{J}, V_r) \cap C^1(\dot{J}, H^{2\alpha_1-2}_{p,\mathcal{B}}) \ , \tag{8.29}$$

from which we obtain, by Lemma 8.1 (i) and condition (7.2), respectively, that

$$A_{\alpha_0-1}(\cdot) \in C^{\alpha_1-r/2}(\dot{J}, \mathcal{L}(H^{2\alpha_0}_{p,\mathcal{B}}, H^{2\alpha_0-2}_{p,\mathcal{B}}))$$

and

$$f \in C(\dot{J}, H^{2\beta_1}_{p,\mathcal{B}}) \ .$$

Since $2\alpha_1 - r > 2\alpha_2 - 2\alpha_0$ and $\beta_1 > \alpha_2 - 1$ we can apply again [13, Theorem 8.2] and the above arguments to show that (8.29) is true with α_1 replaced by α_2. By iterating this reasoning we see finally that u satisfies (7.9), and (7.10) follows by invoking (10) in Theorem 8.2 of [13]. Thus (i) - (iv) and the first assertion of (v) have been proven.

Suppose now that conditions (8.10) and (7.14) are also satisfied. Assume first that $2\alpha_1 < 1 + 1/p$. By replacing $A_{\alpha_1-1}(\cdot)$ by $\omega + A_{\alpha_1-1}(\cdot)$ for a sufficiently large $\omega > 0$, by using (8.19), and by invoking Theorem A.3 of the Appendix we deduce from (8.20) and (8.22) — similarly as in (8.13) - (8.15) — that

$$u \in BC([\tau', \infty), H^{2\alpha_1}_{p,\mathcal{B}}) \quad , \quad \tau' > \tau \ . \tag{8.30}$$

Since $\hat{\sigma}(J \times B, \lambda)$ is bounded in $\mathcal{S}(\Omega)$ by (7.11) and $\pi_p(\hat{\sigma}(J \times B, \lambda))$ is relatively compact in $\mathcal{E}(\Omega)$, it follows from Theorems 5.2 and 5.6 that

$$\omega + A_{\alpha_1 - 1}(\cdot) \in B(J, \mathcal{L}(H^{2\alpha_1}_{p,\mathcal{B}}, H^{2\alpha_1 - 2}_{p,\mathcal{B}})) .$$

Hence we obtain from (8.20), (8.21), and the boundedness of $f(J)$ in $H^{2\beta_0}_{p,\mathcal{B}}$ that

$$u \in BC^1([\tau', \infty), H^{2\alpha_1 - 2}_{p,\mathcal{B}}) \ , \ \tau' > \tau . \tag{8.31}$$

Since (8.30) and (8.31) imply the validity of (8.23) for all $t, t' \geq \tau'$, we see that

$$u \in BUC^{\alpha_1 - r/2}([\tau', \infty), V_r) \ , \ \tau' > \tau . \tag{8.32}$$

Assume now that $2\alpha_1 > 1 + 1/p$. First we observe that (8.25) is now valid for all $t \in J$. Moreover from (8.20) we obtain the representation (8.15), where now U denotes the parabolic fundamental solution for $\{\omega + A_{\alpha_1 - 1}(t) \, ; \, t \in J\}$. Thanks to (8.19) we can invoke again Theorem A.3 of the appendix to derive from the analogue of the representation (8.15) the fact that

$$(\omega + A_{\alpha_1 - 1}(\cdot))u(\cdot) \in B([\tau', \infty), H^{2\alpha_1 - 2}_{p,\mathcal{B}}) \ , \ \tau' > \tau . \tag{8.33}$$

Hence the uniform analogue to (8.25) shows that

$$u \in B([\tau', \infty), H^{2\alpha_1}_p) \ , \ \tau' > \tau . \tag{8.34}$$

Moreover we obtain from (8.20), (8.21), and (8.33) that

$$u \in BC^1([\tau', \infty), H^{2\alpha_1 - 2}_p) . \tag{8.35}$$

Since (8.34) and (8.35) imply (8.27) for all $t, t' \geq \tau'$, we deduce from (8.28) that (8.32) is also true in this case. Since we can replace in these arguments α_1 by a slightly bigger number γ, similarly as above, we obtain finally that

$$u \in BC([\tau', \infty), H^{2\alpha_1}_p) \cap BUC^{\alpha_2 - r/2}([\tau', \infty), V_r) \cap BC^1([\tau', \infty), H^{2\alpha_1 - 2}_{p,\mathcal{B}}) \ , \ \tau' > \tau ,$$

no matter if $2\alpha_1$ is less or greater than $1 + 1/p$. Hence (7.11), (7.12) and Theorem (6.3) imply

$$A_{\alpha_0 - 1}(\cdot) \in BUC^{\alpha_1 - r/2}([\tau', \infty), \mathcal{L}(H^{2\alpha_0}_{p,\mathcal{B}}, H^{2\alpha_0 - 2}_{p,\mathcal{B}})) ,$$

and condition (7.14) gives

$$f \in BC([\tau', \infty), H^{2\beta_1}_{p,\mathcal{B}})$$

for $\tau' > \tau$. Now the assertion follows by iterating these arguments. ∎

9. Classical solvability. In the following we put

$$C^\sigma(\overline{\Omega}) := C^\sigma(\overline{\Omega}, \mathbb{K}^N) \ , \ \sigma \geq 0 ,$$

etc. By a *classical solution of* $(QRDS)_{(\tau, u_0, \lambda)}$ *on* J we mean a function

$$u \in C(J, V) \cap C(\dot{J}, C^1(\overline{\Omega})) \cap C(\dot{J}, C^2(\Omega)) \cap C^1(\dot{J}, C(\overline{\Omega}))$$

which satisfies $(QRDS)_{(\tau, u_0, \lambda)}$ pointwise on J.

Proposition 9.1. *Every classical solution u of $(QRDS)_{(\tau,u_0,\lambda)}$ on J is a solution on J. Thus $u \in C(\dot{J}, H_p^2)$ and $(QRDS)_{(\tau,u_0,\lambda)}$ possesses at most one maximal classical solution.*

Proof: Let $T' \in \dot{J}$ be arbitrary and observe that $u \in C([\tau,T'], H_p^s) \cap C^{1-}([\tau,T'], L_p)$. Hence $u \in C^{1-r/s}([\tau,T'], V_r)$, by interpolation. Now it follows from Lemma 8.1 that

$$[t \mapsto \mathbb{A}_\xi(t) := \mathbb{A}_\xi(t, u(t), \lambda)] \in C^{1-r/s}([\tau,T'], \mathcal{L}(\mathbb{E}_{1+\xi}, \mathbb{E}_\xi))$$

and that $\{\mathbb{A}_\xi(t)\,;\, \tau \le t \le T'\}$ is a regularly bounded subset of $\mathcal{H}(\mathbb{E}_\xi, \mathbb{E}_{1+\xi})$ for $\alpha - 1 \le \xi \le \varepsilon$. Since

$$f := F(\cdot, u(\cdot), \lambda) \in C([\tau,T'], H_{p,\mathcal{B}}^{2\beta_0}) ,$$

as follows from assumption (A3) and the fact that $u \in C([\tau,T'],V)$, and since $H_{p,\mathcal{B}}^{2\beta_0} \doteq \mathbb{E}_\gamma$ with $\gamma > \varepsilon$, we deduce from [13, Theorem 8.2] that the Cauchy problem

$$\dot{v} + \mathbb{A}_\xi(t) v = f(t) \quad , \; \tau < t \le T' \; , \; v(\tau) = u_0 \tag{9.1$_\xi$}$$

has for each $\xi \in [\alpha - 1, \varepsilon]$ a unique solution v_ξ. It is obvious that v_ξ is independent of ξ so that $v_{\alpha-1} = v_0$. On the other hand it is obvious that u is a solution of $(9.1)_{\alpha-1}$. Hence $u = v_{\alpha-1} = v_0$ by uniqueness, which shows that u is the unique solution of $(9.1)_0$. Consequently u is a solution on J of (8.5) which implies that $J \subset J(\tau, u_0, \lambda)$ and $u = u(\cdot,\tau,u_0,\lambda)|J$. This proves the assertion. ∎

Under some mild additional regularity assumptions we shall now show that the converse of Proposition 9.1 is also true. For this we put, for $0 < \varepsilon < 1/2$, any nontrivial interval $J \subset \mathbb{R}$, and any Banach space X

$$C^{(2\varepsilon)}(\overline{\Omega} \times J, X) := B(J, C^{2\varepsilon}(\overline{\Omega}, X)) \cap C^\varepsilon(J, C(\overline{\Omega}, X))$$

and

$$C^{(1+2\varepsilon)}(\overline{\Omega} \times J, X) := B(J, C^{1+2\varepsilon}(\overline{\Omega}, X)) \cap C^\varepsilon(J, C^1(\overline{\Omega}, X)) \cap C^{\varepsilon+1/2}(J, C(\overline{\Omega}, X)) ,$$

respectively. Similar notations are used if $\overline{\Omega}$ is replaced by $\partial\Omega$.

Theorem 9.2. *Suppose that $p > n/2$ and, given $(\tau, u_0, \lambda) \in [0,T) \times V \times \Lambda$, let $u := u(\cdot,\tau,u_0,\lambda)$. Moreover suppose that there exists ε with $0 < 2\varepsilon < (2 - n/p) \wedge 1$ so that*

$$\begin{aligned} (a_{jk}(\cdot, u(\cdot), \lambda)) &\in C^{(1+2\varepsilon)}(\overline{\Omega} \times J', \mathcal{L}(\mathbb{K}^N))^{n^2} , \\ (a_0, a_1, \dots, a_n)(\cdot, u(\cdot), \lambda) &\in C^{(2\varepsilon)}(\overline{\Omega} \times J', \mathcal{L}(\mathbb{K}^N))^{n+1} , \\ b_0(\cdot, u(\cdot), \lambda) &\in C^{(1+2\varepsilon)}(\partial\Omega \times J', \mathcal{L}(\mathbb{K}^N)) , \end{aligned} \tag{9.2}$$

and

$$f := F(\cdot, u(\cdot), \lambda) \in C^{(2\varepsilon)}(\overline{\Omega} \times J', \mathbb{K}^N) , \tag{9.3}$$

for every compact subinterval J' of $\dot{J} := \dot{J}(\tau, u_0, \lambda)$. Then u is a classical solution. Moroever

$$u \in B(J', C^{2+2\varepsilon}(\overline{\Omega})) \cap C^\varepsilon(\dot{J}, C^2(\overline{\Omega})) \cap C^{1+\varepsilon}(\dot{J}, C(\overline{\Omega})) . \tag{9.4}$$

Proof: Put $(\tilde{\mathcal{A}},\tilde{\mathcal{B}})(t) := (\mathcal{A},\mathcal{B})(t,u(t),\lambda)$ for $t \in J$, and let $J' \subset\subset \dot{J}$ be arbitrary. Choose any $\varphi \in \mathcal{D}(\dot{J},\mathbb{R})$ with $\varphi|J' = 1$ and consider the linear parabolic initial boundary value problem

$$\begin{aligned} \dot{v} + \tilde{\mathcal{A}}(t)v &= \tilde{F}(t) && \text{in } \Omega \times \dot{J} \ , \\ \tilde{\mathcal{B}}(t)v &= 0 && \text{on } \partial\Omega \times \dot{J} \ , \\ v(\cdot,0) &= 0 && \text{on } \Omega \ , \end{aligned} \tag{9.5}$$

where $\tilde{F} := \varphi f + \dot{\varphi}u$. Observe that $\tilde{F} \in C^{(2\varepsilon)}(\overline{\Omega} \times J, \mathbb{K}^N)$ since $u \in C^{\varepsilon}(\dot{J}, H_p^{2-2\varepsilon}) \hookrightarrow C^{\varepsilon}(\dot{J}, C(\overline{\Omega}))$, as follows by interpolation from $u \in C(\dot{J}, H_p^2) \cap C^1(\dot{J}, L_p)$, and since $H_p^2 \hookrightarrow C^{2\varepsilon}(\overline{\Omega})$. Hence it follows from [40, Theorem 4.9] that (9.5) possesses a unique classical solution v and

$$v \in B(J, C^{2+2\varepsilon}(\overline{\Omega})) \cap C^{\varepsilon}(J, C^2(\overline{\Omega})) \cap C^{1+\varepsilon}(J, C(\overline{\Omega})) \ . \tag{9.6}$$

It is obvious that v is also the unique solution of

$$\dot{w} + \mathbb{A}(t,u(t),\lambda)w = \tilde{F}(t) \ \ , \ t \in \dot{J} \ , \ w(0) = 0 \ . \tag{9.7}$$

Since φu is a solution of (9.7), it follows that $\varphi u = v$. Now the assertion is a consequence of (9.6) and the arbitrariness of J'. ∎

Corollary 9.3. (Standard Situation): *Let the hypotheses of Propositions* 7.1 *and* 7.2 *be satisfied. Then $u := u(\cdot,\tau,u_0,\lambda)$ is, for each $(\tau,u_0,\lambda) \in [0,T) \times V \times \Lambda$, the unique maximal classical solution of $(QRDS)_{(\tau,u_0,\lambda)}$. Moreover*

$$u \in C(\dot{J}, C^{2+2\varepsilon}(\overline{\Omega})) \cap C^{\varepsilon}(\dot{J}, C^2(\overline{\Omega})) \cap C^{1+\varepsilon}(\dot{J}, C(\overline{\Omega}))$$

for any ε with $0 \le 2\varepsilon < 1 - n/p$.

Proof: Since $u \in C(J', H_p^2) \cap C^1(J', L_p)$ for any $J' \subset\subset \dot{J}$, it follows by interpolation that

$$u \in C^{\sigma}(J', H_p^{2-2\sigma}) \ \ , \ 0 < \sigma < 1 \ .$$

Since $H^{\sigma} \hookrightarrow C^{\sigma - n/p}$ for $\sigma > n/p$, it is easily verified that

$$u \in C^{(1+2\varepsilon)}(\overline{\Omega} \times J', \mathbb{C}^N)$$

for $0 < \varepsilon < 1 - n/p$. It is an easy consequence of the regularity assumptions of Propositions 7.1 and 7.2 that the hypotheses (9.2) and (9.3) are satisfied. Now the assertion follows from the fact that $u \in B(J', C^{2+2\varepsilon}(\overline{\Omega})) \cap C(\dot{J}, C(\overline{\Omega}))$ for every $J \subset\subset \dot{J}$ implies $u \in C(\dot{J}, C^{2+2\varepsilon'}(\overline{\Omega}))$ for $0 \le \varepsilon' < \varepsilon$ (by interpolation, e.g. [43, Theorem 4.5.2.1]) and from Proposition 9.1. ∎

Corollary 9.4. (Standard Situation): *Let the hypotheses of Propositions* 7.1 *and* 7.2 *be satisfied. Let $\lambda \in \Lambda$ be fixed and suppress it from the notation. Assume also that*

$$a_{jk}, a_j, a_0 \in C^{\infty}(\overline{\Omega} \times [0,T] \times G, \mathcal{L}(\mathbb{R}^N)) \ , \quad b_0 \in C^{\infty}(\partial\Omega \times [0,T] \times G, \mathcal{L}(\mathbb{R}^N))$$

and

$$f \in C^\infty(\overline{\Omega} \times [0,T] \times G \times \mathbb{R}^{nN}, \mathbb{R}^N) .$$

Then

$$u := [(x,t) \mapsto u(x,t,\tau,u_0)] \in C^\infty(\overline{\Omega} \times \dot{J})$$

for each $(\tau, u_0) \in [0,T) \times V$, *where* $J := J(\tau, u_0)$.

Proof: Using the notations of [40], the proofs of Theorem 9.2 and Corollary 9.3 show that $u \in C^{\ell,\ell/2}(\overline{\Omega} \times \dot{J})$, where $\ell := 2 + 2\varepsilon$. Suppose now that $u \in C^{m,m/2}(\overline{\Omega} \times \dot{J})$ for $m := \ell + k$, where $k \in \mathbb{N}$. Then the coefficients of $\mathcal{A}(\cdot, u)$ belong to $C^{m-1,(m-1)/2}(\overline{\Omega} \times \dot{J})$, those of $\mathcal{B}(\cdot, u)$ to $C^{m,m/2}(\partial\Omega \times \dot{J})$, and $F(\cdot, u) \in C^{m,m/2}(\overline{\Omega} \times \dot{J})$. Hence, by applying again [40, Theorem 4.9] in the same way as in the proof of Theorem 9.2, we see that $u \in C^{m+1,(m+1)/2}(\overline{\Omega} \times \dot{J})$, which proves the assertion.

10. Smoothness in the H_p^s-topology. We assume first that Λ is a one-point space, i.e., $\Lambda := \{\lambda\}$, and omit any reference to λ. Moreover, we use the notations of Section 8, fix any $u_0 \in V \cap \mathbb{E}_{1+\varepsilon}$, and put $u := u(\cdot, 0, u_0)$. Finally we assume that

$$(\hat{\sigma}(\cdot), F) \in C^{2-}([0,T] \times V_r, \mathcal{S}_p^{\hat{\rho}}(\Omega) \times H_{p,\mathcal{B}}^{2\beta_0}) , \tag{10.1}$$

although it is easily verified that fewer regularity assumptions with respect to the first variable would suffice. Then it is an immediate consequence of Theorem 6.3 that

$$(\mathbb{A}_\xi, F) \in C^{2-}([0,T] \times V_{(\beta)}, \mathcal{L}(\mathbb{E}_{1+\xi}, \mathbb{E}_\xi) \times \mathbb{E}_\xi) \tag{10.2}$$

for $\alpha - 1 \leq \xi \leq \varepsilon$. Thus, given any $T' \in \dot{J}(0, u_0)$,

$$\mathbb{B}_\xi(t) := \mathbb{A}_\xi(t, u(t)) + \partial_2 \mathbb{A}_\xi(t, u(t))[u(t), \cdot] - \partial_2 F(t, u(t))$$

is well defined for $0 \leq t \leq T'$.

Lemma 10.1. *Let* $\tilde{\rho}(\xi) := (\alpha - \beta) \wedge (\varepsilon - \xi)$ *for* $\alpha - 1 \leq \xi < \varepsilon$. *Then*

$$\mathbb{B}_\xi \in C^{\tilde{\rho}(\xi)}([0,T'], \mathcal{H}(\mathbb{E}_\xi, \mathbb{E}_{1+\xi})) .$$

Proof: We know from Lemma 8.2 and [15, Theorem 7.1] that

$$u \in C([0,T'], V) \cap C^{\alpha-\beta}([0,T'], V_{(\beta)}) . \tag{10.3}$$

Moreover (8.1) and the proof of (8.9) show that

$$u \in C^{\varepsilon-\xi}([0,T'], \mathbb{E}_{1+\xi}) . \tag{10.4}$$

Hence we deduce from (10.3) and Lemma 8.1 that

$$\mathbb{A}_\xi(\cdot, u(\cdot)) \in C^{\alpha-\beta}([0,T'], \mathcal{H}(\mathbb{E}_\xi, \mathbb{E}_{1+\xi})) , \tag{10.5}$$

whereas (10.2) and (10.4) imply

$$[t \mapsto \partial_2 \mathbb{A}_\xi(t, u(t))[u(t), \cdot] - \partial_2 F(t, u(t))] \in C^{\tilde{\rho}(\xi)}([0,T'], \mathcal{L}(\mathbb{E}_\beta, \mathbb{E}_\xi)) . \tag{10.6}$$

Since $\mathbb{E}_{1+\xi} \hookrightarrow \mathbb{E}_\beta$ for $\alpha - 1 \leq \xi \leq \varepsilon$, it follows that

$$\mathbb{B}_\xi \in C^{\tilde{\rho}(\xi)}([0,T'], \mathcal{L}(\mathbb{E}_{1+\xi}, \mathbb{E}_\xi)) .$$

Finally we deduce from (8.1), (10.5), (10.6), and a well known perturbation theorem for generators of analytic semigroups (e.g. [32, Theorem 3.2.1]) that $B(t) \in \mathcal{H}(\mathbb{E}_\xi, \mathbb{E}_{1+\xi})$ for $0 \leq t \leq T'$, which proves the assertion. ∎

Consider now the linear Cauchy problem

$$\begin{aligned} \dot{v} + \mathbb{B}_\xi(t)v &= \partial_1 F(t, u(t)) - \partial_1 \mathbb{A}_\xi(t, u(t))u(t) \quad , \; 0 < t \leq T' , \\ v(0) &= -\mathbb{A}_\xi(0, u_0)u_0 + F(0, u_0) . \end{aligned} \tag{10.7}$$

It follows from (10.2) and (10.3) that

$$\partial_1 F(\cdot, u(\cdot)) - \partial_1 \mathbb{A}_\xi(\cdot, u(\cdot)) \in C^{\alpha-\beta}([0,T'], \mathbb{E}_\varepsilon) .$$

Since

$$-\mathbb{A}_\xi(0, u_0)u_0 + F(0, u_0) \in \mathbb{E}_\varepsilon , \tag{10.8}$$

we deduce from Lemma 10.1 and [15, Theorem 5.3] that (10.7) has a unique solution

$$v \in C([0,T'], \mathbb{E}_\varepsilon) \cap C^1((0,T'], \mathbb{E}_\xi) \cap B_{(1+\xi-\varepsilon)(\beta-\xi)} \dot{C}_{T'}(\mathbb{E}_\beta) \cap C((0,T'], \mathbb{E}_{1+\xi}) \tag{10.9}$$

for $\alpha - 1 \leq \xi < \varepsilon$.

Let $h \in (0, T')$ be given and put

$$w_h(t) := u(t+h) - u(t) - v(t)h \quad , \; 0 \leq t \leq T' - h .$$

Then it is not difficult to see that w_h is a solution of the Cauchy problem

$$\begin{aligned} \dot{w} + \mathbb{A}_\xi(t, u(t))w &= C_\xi(t,h)w + f_\xi(t,h)h \quad , \; 0 < t \leq T' - h , \\ w(0) &= g(h)h , \end{aligned} \tag{10.10}$$

where, setting $t(\tau, h) := (t + \tau h, u(t) + \tau(u(t+h) - u(t)))$,

$$C_\xi(t,h) := \int_0^1 \{\partial_2 F(t(\tau,h)) - \partial_2 \mathbb{A}_\xi(t(\tau,h))[u(t+h), \cdot]\} d\tau ,$$

$$\begin{aligned} f_\xi(t,h) := &\int_0^1 \{\partial_1 F(t(\tau,h)) - \partial_1 F(t, u(t)) + [\partial_2 F(t(\tau,h)) - \partial_2 F(t, u(t))]v(t)\} d\tau \\ &- \int_0^1 [\partial_1 \mathbb{A}_\xi(t(\tau,h))u(t+h) - \partial_1 \mathbb{A}_\xi(t, u(t))u(t)] d\tau \\ &- \int_0^1 \{\partial_2 \mathbb{A}_\xi(t(\tau,h))[u(t+h), v(t)] - \partial_2 \mathbb{A}_\xi(t, u(t))[u(t), v(t)]\} d\tau , \end{aligned}$$

and

$$g(h) := \begin{cases} h^{-1}(u(h)-u_0) + \mathbb{A}_\xi(0,u_0)u_0 - F(0,u_0) , & h>0 , \\ 0 & h=0 , \end{cases}$$

respectively. Observe that (10.2) - (10.4) imply

$$C_\xi \in C^{\tilde\rho(\xi)}([0,T''] \times [0,T'-T''], \mathcal{L}(\mathbb{E}_\beta, \mathbb{E}_\xi)) \tag{10.11}$$

for $0 < T'' < T'$. By using also (10.9) it follows similarly that

$$[h \mapsto f_\xi(\cdot,h)] \in C([0,T'-T''], B_{(1+\xi-\varepsilon)(\beta-\xi)}\dot{C}_{T''}(\mathbb{E}_\xi)) . \tag{10.12}$$

Finally,

$$g \in C([0,T'],\mathbb{E}_\varepsilon) , \tag{10.13}$$

due to the fact that $u_0 \in \mathbb{E}_{1+\varepsilon}$. Observe also that

$$f_\xi(\cdot,0) = 0 . \tag{10.14}$$

We fix now $T'' \in (0,T')$ arbitrarily and put

$$\mathbb{C}_\xi(t,h) := \mathbb{A}_\xi(t,u(t)) - C_\xi(t,h)$$

for $0 \le t \le T''$, $0 \le h \le T'-T''$, and $\alpha - 1 \le \xi < \varepsilon$.

Lemma 10.2. *Given* $\xi \in [\alpha-1,\varepsilon)$,

$$[h \mapsto \mathbb{C}_\xi(\cdot,h)] \in C([0,T'-T''], C^{\tilde\rho(\xi)}([0,T''],\mathcal{H}(\mathbb{E}_\xi,\mathbb{E}_{1+\xi}))) .$$

Proof: This follows from (10.5), (10.11), and the perturbation theorem for generators of analytic semigroups, used already in the proof of Lemma 10.1. ∎

Fix $\zeta \in (0,\varepsilon)$, put $\vartheta := (1+\zeta-\varepsilon)(\beta-\zeta)$ and observe that $\varepsilon + \vartheta < 1+\zeta$. Hence it follows from (10.12), (10.13), Lemma 10.2, and [15, Theorem 5.3] that the linear Cauchy problem

$$\dot w + \mathbb{C}_0(t,h)w = f_\zeta(t,h)h, \quad w(0) = g(h)h, \quad 0 < t < T'' , \tag{10.15}$$

possesses a unique solution

$$w(\cdot,h) \in C([0,T''],\mathbb{E}_\varepsilon) \cap C^1((0,T''],\mathbb{E}_0) \cap B_{1-\varepsilon}\dot{C}_{T''}(\mathbb{E}_1)$$

for each $h \in [0,T'-T'']$. It is an easy consequence of Lemma 10.2 and [15, Proposition 4.2] that $\{\mathbb{C}_0(\cdot,h)\,;\, 0 \le h \le T'-T''\}$ is regularly bounded in $C^{\tilde\rho(0)}\big([0,T''],\mathcal{H}(\mathbb{E}_0,\mathbb{E}_1)\big)$. Hence we deduce from [15, Theorem 5.3] and from (10.12) - (10.14) that

$$w(\cdot,h)h^{-1} \to 0 \quad \text{in} \quad B\dot{C}_{T''}(\mathbb{E}_\varepsilon)$$

as $h \to 0$. Since w_h is also a solution of (10.15), thanks to (10.9) and (10.10), it follows that $w(\cdot,h) = w_h$. Hence the right derivative of u on $(0,T'')$ with respect to the topology of $\mathbb{E}_\varepsilon$ equals v. Since $v \in C([0,T''],\mathbb{E}_\varepsilon)$, it follows that v is in fact the derivative of u on $[0,T'']$ with respect to the topology of $\mathbb{E}_\varepsilon$, and (10.9) implies

$$u \in C^1((0,T''],\mathbb{E}_{1+\xi}) \ , \ 0 \le \xi < \varepsilon .$$

In summary, we have essentially proven the following

Theorem 10.3. *Let $\lambda \in \Lambda$ be fixed and omit any reference to it. Suppose that*

$$(\hat{\sigma}, F) \in C^{2-}([0,T] \times V_r, \mathcal{S}_p^{\hat{\rho}}(\Omega) \times H_{p,\mathcal{B}}^{2\beta_0}) .$$

Then, given any $(\tau, u_0) \in [0,T) \times V$ and σ with $s \leq 2\sigma < 1+\rho$,

$$u := u(\cdot, \tau, u_0) \in C^1(\dot{J}, H_{p,\mathcal{B}}^{2\sigma}) ,$$

where $J := J(\tau, u_0)$. Given $\tau' \in \dot{J}$, let $J' := [\tau', \infty) \cap J$. Then $\dot{u}|\dot{J}'$ is the unique solution of the linearized Cauchy problem in $H_{p,\mathcal{B}}^{2\sigma-2}$

$$\begin{aligned} \dot{v} + A_{\sigma-1}(t)v &= -\,\partial_1 A_{\sigma-1}(t)u(t) - \partial_2 A_{\sigma-1}(t)[u(t), v] \\ &\quad + \partial_1 F(t) + \partial_2 F(t)v \quad , \ t \in \dot{J}' , \qquad (10.16) \\ v(\tau') &= -\,A_{\sigma-1}(\tau')u(\tau') + F(\tau') , \end{aligned}$$

where $(A_{\sigma-1}, F)(t) := (A_{\sigma-1}, F)(t, u(t))$ etc.

Proof: Since $u(\tau') \in \mathbb{E}_{1+\varepsilon}$ for $\tau' \in \dot{J}$, the assertion is an obvious consequence of the preceeding considerations. ▮

We assume now that

$$\Lambda \text{ is a nonempty open subset of some Banach space } \Lambda.$$

We put

$$\overset{\circ}{\mathcal{D}}(\tau) := \{(t, u_0, \lambda) \in \mathcal{D}(\tau)\,;\, t > \tau\} \quad , \ \tau \in [0,T) .$$

Then we can prove the following *differentiability theorem.*

Theorem 10.4. *Suppose that*

$$(\hat{\sigma}, F) \in C^{2-}([0,T] \times V_r \times \Lambda, \mathcal{S}_p^{\hat{\rho}}(\Omega) \times H_{p,\mathcal{B}}^{2\beta_0}) . \qquad (10.17)$$

Then

$$u(\cdot, \tau, \cdot, \cdot) \in C^1(\overset{\circ}{\mathcal{D}}(\tau), V) \quad , \ \tau \in [0,T) .$$

Moreover, given $(\tau, u_0, \lambda, h, k) \in [0,T) \times V \times \Lambda \times H_{p,\mathcal{B}}^s \times \mathbf{\Lambda}$, the functions $\partial_3 u(\cdot, \tau, u_0, \lambda)h$ and $\partial_4 u(\cdot, \tau, u_0, \lambda)k$ are the unique solutions on $J := J(\tau, u_0, \lambda)$ of the linearized Cauchy problems in $H_{p,\mathcal{B}}^{2\sigma}$, $s \leq 2\sigma < 1+\rho$:

$$\dot{v} + A_{\sigma-1}(t)v = -\partial_2 A_{\sigma-1}(t)[u(t), v] + \partial_2 F(t)v, \quad v(\tau) = h, \quad t \in \dot{J},$$

and

$$\begin{aligned} \dot{w} + A_{\sigma-1}(t)w &= -\partial_3 A_{\sigma-1}(t)[u(t), k] - \partial_2 A(t)[u(t), w] \\ &\quad + \partial_3 F(t)k + \partial_2 F(t)w, \ \ t \in \dot{J} , \\ w(\tau) &= k , \end{aligned}$$

respectively, where $(A_{\sigma-1}, F)(t) := (A_{\sigma-1}, F)(t, u(t), \lambda)$ etc.

Proof: It follows from (10.17) and Theorem 6.3 that

$$(\mathbb{A}_\xi, F) \in C^{2-}([0,T] \times V_{(\beta)} \times \Lambda, \mathcal{L}(\mathbb{E}_{1+\xi}, \mathbb{E}_\xi) \times \mathbb{E}_\xi)$$

for $\alpha - 1 \leq \xi \leq \varepsilon$. Hence the assertion is a consequence of Theorem 10.3 and [15, Theorem 11.1 and Corollary 11.2]. ▮

For simplicity we have imposed, in (10.17), more regularity restrictions than really needed. We leave it to the reader to weaken that hypothesis.

Theorem 10.5. *Suppose that* $\mathbb{K} = \mathbb{R}$ *and*

$$(\hat{\sigma}, F) \in C^{k+1-}([0,T] \times V_r \times \Lambda, \mathcal{S}_p^{\hat{\rho}}(\Omega) \times H_{p,\mathcal{B}}^{2\beta_0})$$

for some $k \in \mathbb{N}^* \cup \{\infty\}$. *Then*

$$u(\cdot, \tau, \cdot, \cdot) \in C^k(\overset{\circ}{\mathcal{D}}(\tau), V) \quad , \quad \tau \in [0,T) \ ,$$

and the various derivatives of u *are solutions of the linearized Cauchy problems in* $H_{p,\mathcal{B}}^{2\sigma-2}, 1 \leq 2\sigma < 1+\rho$, *which are obtained by differentiating*

$$\dot{u} + A_{\sigma-1}(t, u(t), \lambda)u = F(t, u(t), \lambda) \quad , \quad t \in \dot{J} \ , \ u(\tau) = u_0$$

appropriately repeatedly with respect to t, u, *and* λ, *respectively.*

Proof: Our assumption and Theorem 6.3 imply that

$$(\mathbb{A}_\xi, F) \in C^{k+1-}([0,T] \times V_{(\beta)} \times \Lambda, \mathcal{L}(\mathbb{E}_{1+\xi}, \mathbb{E}_\xi) \times \mathbb{E}_\xi)$$

for $\alpha - 1 \leq \xi \leq \varepsilon$. Hence the assertion concerning the dependence with respect to $(u_0, \lambda) \in V \times \Lambda$ follows from [15, Theorem 11.3].

We fix now $\lambda \in \Lambda$ and suppress it from the notation. By replacing τ by $\tau' \in \dot{J}$, shifting τ then to zero, and by making T smaller, if necessary, we can assume that

$$u \in C^1([0,T], \mathbb{E}_{1+\xi}) \quad , \quad \alpha - 1 < \xi < \varepsilon \ .$$

Hence, denoting the right hand side of the differential equation in (10.16) by $G(t,v) := G_0(t) + G_1(t)v$, it follows that

$$\dot{G}_0 + \dot{G}_1 \dot{u} \in C([0,T], \mathbb{E}_\xi) \quad , \quad \alpha - 1 \leq \xi < \varepsilon \ .$$

Moreover,

$$G_1 \in C^{\bar{\rho}(\xi)}([0,T], \mathcal{L}(\mathbb{E}_\beta, \mathbb{E}_\xi)) \quad , \quad \alpha - 1 \leq \xi < \varepsilon \ ,$$

thanks to (10.6). This implies, similarly as in the proof of Theorem 10.3, that the linear Cauchy problem

$$\begin{aligned} \dot{w} + \mathbb{A}_\xi(t, u(t))w =& - \partial_1 \mathbb{A}_\xi(t, u(t))\dot{u}(t) - \partial_2 \mathbb{A}_\xi(t, u(t))[\dot{u}(t), \dot{u}(t)] \\ &+ \dot{G}_0(t) + \dot{G}_1(t)\dot{u}(t) + G_1(t)w \quad , \quad 0 < t \leq T \ , \\ w(0) =& - \mathbb{A}_\xi(0, u_0)\dot{u}(0) + G(0, \dot{u}(0)) \end{aligned}$$

has a unique solution. From this we deduce, similarly as above, that $w = \dot{v}$, which shows that $\dot{v} = \ddot{u} \in C(\dot{J}, \mathbb{E}_{1+\varepsilon})$. Hence $u \in C^2(\dot{J}, H_{p,\mathcal{B}}^{2\sigma})$ for $1 \leq 2\sigma < 1+\rho$. By induction we obtain that $u \in C^k(\dot{J}, H_{p,\mathcal{B}}^{2\sigma})$ for $1 \leq 2\sigma < 1+\rho$. Since, thanks to [15, Corollary 11.2], the differentiability with respect to (u_0, λ) is uniform with respect to t in compact subintervals of $\dot{J}$, the assertion follows. Details are left to the reader. ∎

Remark 10.6. (Standard Situation): Let the hypotheses of Proposition 7.1 be satisfied and assume, in addition, that

$$a_{jk}, a_j, a_0 \in C^\infty(\overline{\Omega} \times [0,T] \times G \times \Lambda, \mathcal{L}(\mathbb{R}^N)) ,$$
$$b_0 \in C^\infty(\partial\Omega \times [0,T] \times G \times \Lambda, \mathcal{L}(\mathbb{R}^N))$$

and

$$f \in C^\infty(\overline{\Omega} \times [0,T] \times G \times \Lambda, \mathbb{R}^N) .$$

Moreover, denote by $\hat{\sigma}(\cdot)$ the substitution map of Proposition 7.1, by F the substitution map induced by f, and let $s = 1$. Then

$$(\hat{\sigma}, F) \in C^\infty([0,T] \times V_r \times \Lambda, \mathcal{S}_p^{\hat{\rho}}(\Omega) \times H_{p,\mathcal{B}}^{2\beta_0}) .$$

In the more general case that f depends also nonlinearly on ∂u one obtains a similar result provided the derivatives of f satisfy appropriate growth restrictions. We leave it to the reader to prove these facts and to formulate conditions guaranteeing C^k-differentiability.

11. Nonhomogeneous boundary conditions. We close this paper by indicating how *quasilinear reaction-diffusion systems with nonhomogeneous boundary conditions* can be reduced to the situations studied above. For this we consider problems of the form

$$\begin{aligned} \partial_t u + \mathcal{A}(t,u,\lambda)u &= F(t,u,\lambda) && \text{in } \Omega \times (0,T] , \\ \mathcal{B}(t,u,\lambda)u &= G(t,u,\lambda) && \text{on } \partial\Omega \times (0,T] , \\ u(\cdot,0) &= u_0 && \text{on } \Omega . \end{aligned} \tag{11.1}$$

We assume that 'the boundary conditions depend only upon the trace of u', that is,

$$\mathcal{B}(t,u,\lambda) = \mathcal{B}(t,\gamma u,\lambda) , \tag{11.2}$$

and

$$G(t,u,\lambda) = G(t,\gamma u,\lambda) , \tag{11.3}$$

and that

$$(1-\delta)G(t,u,\lambda) = (1-\delta)g(t,\lambda) . \tag{11.4}$$

We try to represent u in the form

$$u = v + w \quad , \quad v \in H_{p,\mathcal{B}}^s , \tag{11.5}$$

where v and w are appropriately chosen. Since the boundary condition in (11.1) and (11.4) imply the 'compatibility condition'

$$(1-\delta)\gamma u = (1-\delta)g ,$$

it follows from (11.5) that

$$\gamma u = \delta\gamma v + \delta\gamma w + (1-\delta)g . \tag{11.6}$$

Suppose that we can determine $w = w(t,u,\lambda)$ so that

$$\mathcal{B}(t,u,\lambda)w = G(t,u,\lambda) \quad , \ \delta\gamma w = 0 \ . \tag{11.7}$$

Then it follows from (11.2), (11.3), and (11.6) that

$$w = w(t, \delta\gamma v + (1-\delta)g(t,\lambda),\lambda) =: \tilde{w}(t,v,\lambda) \ .$$

Put

$$\tilde{\mathcal{A}}(t,v,\lambda) := \mathcal{A}(t, v+\tilde{w}(t,v,\lambda),\lambda) \ , \quad \tilde{\mathcal{B}}(t,v,\lambda) := \mathcal{B}(t,\delta\gamma v + (1-\delta)g(t,\lambda),\lambda) \ ,$$

and

$$\tilde{F}(t,v,\lambda) := F(t, v + w(t,v,\lambda),\lambda) - \tilde{\mathcal{A}}(t,v,\lambda)w(t,v,\lambda) - \partial_t w(t,v,\lambda)$$

and consider the initial boundary value problem

$$\begin{aligned} \partial_t v + \tilde{\mathcal{A}}(t,v,\lambda)v &= \tilde{F}(t,v,\lambda) && \text{in } \Omega \times (0,T] \ , \\ \tilde{\mathcal{B}}(t,v,\lambda)v &= 0 && \text{on } \partial\Omega \times (0,T] \ , \\ v(\cdot,0) &= v_0 && \text{on } \Omega \ , \end{aligned} \tag{11.8}$$

where $v_0 := u_0 - w(0,u_0,\lambda)$. Then we see that problem (11.1) is equivalent to problem (11.8), provided the latter is well defined in the sense considered in the previous sections. However if G satisfies appropriate regularity assumptions and if

$$a_{jk}(t,\delta\gamma v + (1-\delta)g(t,\lambda),\lambda)\nu^j\nu^k \in C(\partial\Omega, \mathcal{G}L(\mathbb{K}^N)) \ ,$$

we can apply Theorem B.3 of the Appendix to represent w in the form

$$w = \mathcal{R}(t,v,\lambda)G(t,\delta\gamma v + (1-\delta)g(t,\lambda),\lambda) \ ,$$

where $\mathcal{R}$ depends appropriately smoothly upon its arguments.

These considerations show that we can reduce problem (11.1) to a problem of the form (11.8), which has been thoroughly studied above, provided conditions (11.2) - (11.4) are satisfied and G has appropriate smoothness properties. Details can be left to the reader.

Appendix.

A. Estimates for evolution operators. It is the purpose of this section to extend the estimates of [13] for parabolic fundamental solutions in interpolation and extrapolation spaces from bounded intervals $[0,T]$ to all of $\mathbb{R}^+$. For this we use the notations of [13].

We impose assumption (**AA**):

X_0 and X_1 are Banach spaces with $X_1 \overset{d}{\hookrightarrow} X_0$.

Each $A(t), t \in \mathbb{R}^+$, is a closed linear operator in X_0 with $D(A(t)) = X_1$.

There exist constants $\rho \in (0,1), \vartheta \in (0,\pi/2), L, M,$ and N so that $\rho(-A(t)) \supset \Sigma_\vartheta$,

$$\|(\lambda + A(t))^{-1}\|_{\mathcal{L}(X_0)} \le M/(1+|\lambda|) \quad , \ \lambda \in \Sigma_\vartheta \ ,$$

$$\|A(t)\|_{\mathcal{L}(X_1,X_0)} + \|A^{-1}(t)\|_{\mathcal{L}(X_0,X_1)} \le N \ ,$$

and

$$\|A(s) - A(t)\|_{\mathcal{L}(X_1,X_0)} \le L|s-t|^\rho$$

for $s,t \in \mathbb{R}^+$.

In addition we impose also assumption $(\mathbf{AX}_\gamma)$:

X_γ is a Banach space with $X_1 \hookrightarrow X_\gamma \hookrightarrow X_0$.
There exist constants K and $0 < \gamma_- \le \gamma_+ < \rho$ so that

$$(1+|\lambda|)\|(\lambda+A(t))^{-1}\|_{\mathcal{L}(X_\gamma)} + |\lambda|^{1-\gamma_+}\|(\lambda+A(t))^{-1}\|_{\mathcal{L}(X_0,X_\gamma)}$$
$$+ |\lambda|^{\gamma_-}\|(\lambda+A(t))^{-1}\|_{\mathcal{L}(X_\gamma,X_1)} \le K$$

for $\lambda \in \Sigma_\vartheta$ and $t \in \mathbb{R}^+$.
$A_\gamma(t)$, the X_γ-realization of $A(t)$, is densely defined for $t \in \mathbb{R}^+$.

We denote by U the parabolic fundamental solution for $\{A(t)\,;\,t \in \mathbb{R}^+\}$ (that is, U restricts to the parabolic fundamental solution for $\{A(t)\,;\,0 \le t \le T\}$ for every $T > 0$), and U_γ is the X_γ-realization of U. Moreover,

$$\mathcal{M} := \{K, L, M, N, \gamma_+, \gamma_-, \rho, \vartheta\}\ .$$

THEOREM A.1. *U_γ is the parabolic fundamental solution for $\{A_\gamma(t)\,;\,t \in \mathbb{R}^+\}$. It possesses X_1 as regularity subspace and satisfies*

$$A_\gamma U_\gamma A_\gamma^{-1} \in C(T_\Delta, \mathcal{L}_s(X_\gamma))\ \ ,\ T > 0\ .$$

There exist a positive constant $\omega(\mathcal{M})$ and, for each $\varepsilon > 0$, a positive constant $c(\varepsilon,\mathcal{M})$ so that

$$\begin{aligned}&\|U(t,s)\|_{\mathcal{L}(X_\alpha)} + \|A_\gamma U_\gamma A_\gamma^{-1}(t,s)\|_{\mathcal{L}(X_\gamma)}\\ &\quad + (t-s)\|U(t,s)\|_{\mathcal{L}(X_0,X_1)} + (t-s)\|A_\beta U_\beta(t,s)\|_{\mathcal{L}(X_\beta)}\\ &\quad + (t-s)^{\gamma_+}\|U(t,s)\|_{\mathcal{L}(X_0,X_\gamma)} + (t-s)^{1-\gamma_-}\|U(t,s)\|_{\mathcal{L}(X_\gamma,X_1)}\\ &\le c(\varepsilon,\mathcal{M})e^{(\omega(\mathcal{M})+\varepsilon)(t-s)}\end{aligned} \tag{A.1}$$

for $0 \le s < t < \infty$, $\alpha \in \{0,\gamma,1\}$, and $\beta \in \{0,\gamma\}$. The constants $\omega(\mathcal{M})$ and $c(\varepsilon,\mathcal{M})$ depend on the indicated quantities, but neither on the individual operatos $A(t)$ nor on $s,t \in \mathbb{R}^+$. Moreover $\omega(\mathcal{M})$ is a continuous function of L which goes to zero as L does.

Proof: If we fix any $T > 0$ and restrict (t,s) to T_Δ, the assertion is precisely the one of [13, Theorem 2.2], except that we have now a more precise (t,s)-dependent estimate for the constant occuring on the right hand side of (A.1). Hence a proof of the theorem will be a modification of the proof of [13, Theorem 2.2], which consists essentially in keeping track of the T-dependence of the various constants occuring in the latter proof. For this reason we will indicate only the most important changes, which have to be made, and leave the routine checking of the remaining steps to the reader.

Let X and Y be Banach spaces, let $T > 0$ be fixed, and let $\sigma \in \mathbb{R}$. Given $f \in C(\dot{T}_\Delta, \mathcal{L}(X,Y))$, put

$$f_{[\sigma]}(t,s) := e^{-\sigma(t-s)}f(t,s)\ .$$

Then

$$(f * g)_{[\sigma]} = f_{[\sigma]} * g_{[\sigma]} \tag{A.2}$$

whenever $f * g$ is well defined.

Suppose that $f \in K(X, 1-\alpha)$ for some $\alpha > 0$. Then it follows from [10, (1.11)] that

$$g := \sum_{j=1}^{\infty} \underbrace{f * \ldots * f}_{j} \in K(X, 1-\alpha)$$

and that

$$\|g(t,s)\|_{\mathcal{L}(X)} \le (t-s)^{\alpha-1}\|f\|_{(1-\alpha)}\Gamma(\alpha)m_{1-\alpha}((t-s)^\alpha\|f\|_{(1-\alpha)}\Gamma(\alpha))\ \ ,\ 0 \le s < t \le T\ ,$$

where

$$m_{1-\alpha}(\xi) := \sum_{j=1}^{\infty} \frac{\xi^{j-1}}{\Gamma(j\alpha)} \quad , \ \xi > 0 \ .$$

By means of Stirling's estimate for the gamma function one can show (cf. [45, p. 6]) that

$$m_{1-\alpha}(\xi) \le c(\alpha) e^{2\xi^{1/\alpha}} \quad , \ \xi > 0 \ ,$$

where, here and in the following, $c, c(\alpha)$, etc. are constants which are independent of T. Consequently,

$$\|g(t,s)\|_{\mathcal{L}(X)} \le c(\alpha)(t-s)^{\alpha-1} \|f\|_{(1-\alpha)} e^{2(\Gamma(\alpha)\|f\|_{(1-\alpha)})^{1/\alpha}(t-s)} \quad , \ 0 \le s < t \le T \ . \tag{A.3}$$

We return now to the proof of [13, Theorem 2.2]. Our assumptions imply that

$$\|k\|_{K(X_\alpha, X_0, 1-\alpha-\rho)} + \|k\|_{K(X_\gamma, X_0, 1-\gamma_- -\rho)} \le c(\mathcal{M}_0) L \quad , \ \alpha \in \{0,1\} \ , \tag{A.4}$$

where $\mathcal{M}_0 := \mathcal{M} \backslash \{L\}$. Since

$$w = k + k * w = \sum_{j=1}^{\infty} \underbrace{k * \ldots * k}_{j} \ ,$$

we deduce from (A.3) and (A.4) that

$$\|w(t,s)\|_{\mathcal{L}(X_0)} \le c(\mathcal{M})(t-s)^{\rho-1} e^{\sigma(t-s)} \quad , \ 0 \le s < t < \infty \ ,$$

where

$$\sigma := \sigma(\mathcal{M}) = 2(\Gamma(\rho) c(\mathcal{M}_0) L)^{1/\rho} > 0 \ . \tag{A.5}$$

Hence

$$\|w_{[\sigma]}\|_{K(X_0, 1-\rho)} \le c(\mathcal{M}) \ . \tag{A.6}$$

Observe that (A.2) implies

$$U_{[\sigma]} = a_{[\sigma]} + a_{[\sigma]} * w_{[\sigma]} \ . \tag{A.7}$$

From our assumptions we deduce also that the norm of $a_{[\sigma]}$ in

$$K(X_\alpha, 0) \cap K(X_0, X_1, 1) \cap K(X_0, X_\gamma, \gamma_+) \cap K(X_\gamma, X_1, 1-\gamma_-)$$

for $\alpha \in \{0, \gamma, 1\}$ is bounded by $c(\mathcal{M})$. Using these facts and (A.6) we obtain from (A.7) and (A.2) that

$$\begin{aligned} &\|U_{[\sigma]}(t,s)\|_{\mathcal{L}(X_\alpha)} + (t-s)^{\gamma_+} \|U_{[\sigma]}(t,s)\|_{\mathcal{L}(X_0, X_\gamma)} + (t-s)^{1-\gamma_-} \|U_{[\sigma]}(t,s)\|_{\mathcal{L}(X_\gamma, X_1)} \\ &\quad \le c(\mathcal{M})(1 + (t-s)^\rho) \end{aligned} \tag{A.8}$$

for $0 \le s < t < \infty$ and $\alpha \in \{0, \gamma\}$.

Next we observe that

$$\partial_1 U_{[\sigma]} = -(\sigma + A) U_{[\sigma]} = \partial_1 a_{[\sigma]} + \partial_1 (a_{[\sigma]} * w_{[\sigma]})$$

and that

$$\partial_1 a_{[\sigma]} = -(\sigma + A) a_{[\sigma]} = -\sigma a_{[\sigma]} - A a \, e^{-\sigma(t-s)} \ .$$

Using these facts and going through Step (ii) of the proof of [13, Theorem 2.2], it is not difficult to see that

$$(t-s)\{\|AU_{[\sigma]}(t,s)\|_{\mathcal{L}(X_\alpha)} + \|U_{[\sigma]}(t,s)\|_{\mathcal{L}(X_0, X_1)}\} \le c(\mathcal{M}) p(t-s) \tag{A.9}$$

for $0 \le s < t < \infty$ and $\alpha \in \{0, \gamma\}$, where, here and in the following, p denotes a suitable polynomial with positive coefficients.

In Step (iii) we replace b and h by $b_{[\sigma]}$ and $h_{[\sigma]}$, respectively. Then the norms of $Ah_{[\sigma]}$ in $K(X_1, X_\gamma, 1+\gamma_+ - \rho)$ and of $Ah_{[\sigma]}A^{-1}$ in $K(X_0, X_\gamma, 1+\gamma_+ - \rho)$ can be estimated by $c(\mathcal{M})$. Moreover,

$$\|Ab_{[\sigma]}A^{-1}(t,s)\|_{\mathcal{L}(X_\gamma)} \leq c(\mathcal{M})p(t-s) \quad , \ 0 \leq s < t < \infty \ .$$

Hence, given any $\varepsilon > 0$,

$$\|Ab_{[\sigma+\varepsilon/2]}A^{-1}\|_{K(X_\gamma,0)} \leq c(\varepsilon, \mathcal{M}) \ . \tag{A.10}$$

Observe that

$$AU_{[\sigma+\varepsilon/2]}A^{-1} = Ab_{[\sigma+\varepsilon/2]}A^{-1} + r * Ab_{[\sigma+\varepsilon/2]}A^{-1} \ ,$$

where

$$r := \sum_{j=1}^{\infty} \underbrace{d * \ldots * d}_{j} \quad , \ d := Ah_{[\sigma+\varepsilon/2]}A^{-1} \ ,$$

and that the norm of d in $K(X_\gamma, 1+\gamma_+ - \rho)$ is bounded by $c(\mathcal{M})$. Hence it follows from (A.3) that

$$\|r(t,s)\|_{\mathcal{L}(X_\gamma)} \leq c(\mathcal{M})(t-s)^{\rho-\gamma_+-1}e^{\sigma_1(t-s)} \quad , \ 0 \leq s < t < \infty \ , \tag{A.11}$$

where

$$\sigma_1 := \sigma_1(\mathcal{M}) := c(\mathcal{M}_0)L^{1/(\rho-\gamma_+)} > 0 \ . \tag{A.12}$$

Thanks to (A.2),

$$AU_{[\sigma+\sigma_1+\varepsilon/2]}A^{-1} = Ab_{[\sigma+\sigma_1+\varepsilon/2]}A^{-1} + r_{[\sigma_1]} * Ab_{[\sigma+\sigma_1+\varepsilon/2]}A^{-1} \ , \tag{A.13}$$

and it follows from (A.10) that

$$\|Ab_{[\sigma+\sigma_1+\varepsilon/2]}A^{-1}\|_{K(X_\gamma,0)} \leq c(\varepsilon, \mathcal{M}) \ . \tag{A.14}$$

Using (A.11), (A.13), and (A.14) we find that

$$\|AU_{[\sigma+\sigma_1+\varepsilon/2]}A^{-1}\|_{\mathcal{L}(X_\gamma)} \leq c(\varepsilon, \mathcal{M})p(t-s) \quad , \ 0 \leq s < t < \infty \ . \tag{A.15}$$

Recall that $U_{[\sigma]} = b_{[\sigma]} + h_{[\sigma]} * U_{[\sigma]}$ implies

$$U_{[\sigma]} = b_{[\sigma]} + e_{[\sigma]} * b_{[\sigma]} \ , \tag{A.16}$$

where

$$e_{[\sigma]} := \sum_{j=1}^{\infty} \underbrace{h_{[\sigma]} * \ldots * h_{[\sigma]}}_{j} \ .$$

Since the norms of $b_{[\sigma]}$ in $K(X_1, 0)$ and of $h_{[\sigma]}$ in $K(X_1, 1-\rho)$, respectively, are bounded by $c(\mathcal{M})$, we deduce from (A.3) that

$$\|e_{[\sigma]}(t,s)\|_{\mathcal{L}(X_1)} \leq c(\mathcal{M})(t-s)^{\rho-1}e^{\sigma_2(t-s)} \quad , \ 0 \leq s < t < \infty \ .$$

where

$$\sigma_2 := \sigma_2(\mathcal{M}) := c(\mathcal{M}_0)L^{1/\rho} > 0 \ . \tag{A.17}$$

Using these facts it follows from (A.16) that

$$\|U_{[\sigma+\sigma_2]}(t,s)\|_{\mathcal{L}(X_1)} \leq c(\mathcal{M})p(t-s) \quad , \ 0 \leq s < t < \infty \ . \tag{A.18}$$

Finally, letting $\omega := \omega(\mathcal{M}) := \sigma + \sigma_1 + \sigma_2$, the assertion is an easy consequence of (A.8), (A.9), (A.16), and (A.18), as well as of (A.5), (A.12), and (A.17). ∎

LEMMA A.2. *Let assumption* (AA) *be satisfied, fix any* $\gamma \in (0, \rho)$, *and put* $X_\gamma := [X_0, X_1]_\gamma$. *Then assumption* (AX_γ) *is satisfied with* $\gamma_+ := \gamma_- := \gamma$ *and* $K := K(M, N, \gamma)$.

Proof: This is an easy consequence of the properties of the complex interpolation functor (cf. [9, Appendix]). ∎

After these preparations we can prove the main result of this section.

THEOREM A.3. *Let Assumption* (AA) *be satisfied, suppose that* $0 < \alpha < \gamma < 1$ *and that* $\alpha < \rho$, *and let* $X_\xi := [X_0, X_1]_\xi$ *for* $\xi \in \{\alpha, \gamma\}$. *Moreover put* $\mathcal{P} := \{L, M, N, \alpha, \gamma, \rho, \vartheta\}$. *Then there exist a positive constant* $\omega(\mathcal{P})$ *and, given any* $\varepsilon > 0$, *a constant* $c(\varepsilon, \mathcal{P})$ *so that*

$$(t-s)^{\eta-\alpha}\|U(t,s)\|_{\mathcal{L}(X_\alpha, X_\eta)} + (t-s)^{1+\alpha-\gamma}\|AU(t,s)\|_{\mathcal{L}(X_\gamma, X_\alpha)} \le c(\varepsilon, \mathcal{P})e^{(\omega(\mathcal{P})+\varepsilon)(t-s)}$$

for $\eta \in \{\gamma, 1\}$ *and* $0 \le s < t < \infty$. *Moreover,* $\omega(\mathcal{P})$ *is continuous in* L *and tends to zero as* L *does.*

Proof: From Theorem A.1 and Lemma A.2 we obtain easily the estimate

$$(t-s)^{\eta-\alpha}\|U(t,s)\|_{\mathcal{L}(X_\alpha, X_\eta)} \le c(\varepsilon, \mathcal{P})e^{(\omega(\mathcal{P})+\varepsilon)(t-s)} \quad , \ 0 \le s < t < \infty \ ,$$

by interpolation. The fact that $(t-s)^{1+\alpha-\gamma}\|AU(t,s)\|_{\mathcal{L}(X_\gamma, X_\alpha)}$ has an estimate of the same form follows by using Lemma A.2 and modifying the proof of [13, Lemma 8.1] along the lines of the proof of Theorem A.1. This proves the assertion. ∎

B. Extension of boundary values. Let E be a Banach space and suppose that $-\Lambda \in \mathcal{H}(E)$ with *type* $\Lambda < 0$. Given $m \in \mathbb{N}$, we define Banach spaces

$$E_m := E_m(\Lambda) := (D(\Lambda^m), \|\Lambda^m \cdot\|)$$

and

$$W_q^m(E) := \bigcap_{j=0}^{m} W_q^j((0,\infty), E_{m-j}) \quad , \ 1 \le q \le \infty \ ,$$

where $W_q^j((0,\infty), E_\ell)$ are the usual Sobolev spaces of E_ℓ-valued distributions on $(0,\infty)$.

We denote by $(\cdot,\cdot)_{\theta,q}$, $0 < \theta < 1$, $1 \le q \le \infty$, the standard real interpolation spaces.

We fix $p \in (1,\infty)$ and put

$$E_{m+\vartheta} := (E_m, E_{m+1})_{\vartheta,p} \quad , \ \vartheta := 1 - 1/p \ , \ m \in \mathbb{N} \ .$$

It follows that

$$\Lambda^k \in \mathcal{L}(E_{s+k}, E_s) \quad , \ k \in \mathbb{N} \ , \ s \in \mathbb{N} \cup (\vartheta + \mathbb{N}) \ , \tag{B.1}$$

and that there exists a constant $\omega > 0$ so that

$$\|e^{t\Lambda}\|_{\mathcal{L}(E_r, E_s)} \le c_{r,s}t^{r-s}e^{-\omega t} \quad , \ t > 0 \ , \ r \le s \ , \ r, s \in \mathbb{N} \cup (\vartheta + \mathbb{N}) \tag{B.2}$$

(cf. [9, Theorem 10]).

PROPOSITION B.1. *Suppose that*

$$(E, E_m)_{\vartheta/m} \doteq E_\vartheta \quad , \ m \in \mathbb{N}^* \ . \tag{B.3}$$

Given $k \in \mathbb{N}$, *define*

$$R_k \in \mathcal{L}(E, BC(\mathbb{R}^+, E)) \tag{B.4}$$

by

$$R_k g(t) := (1/k!)t^k e^{t\Lambda}g \quad , \ t \ge 0 \ , \ g \in E \ . \tag{B.5}$$

Then

$$R_k \in \bigcap_{m=0}^{\infty} \mathcal{L}(E_{m+\vartheta}, W_p^{m+1+k}(E)) \ . \tag{B.6}$$

Moreover

$$R_k \in \bigcap_{m=0}^{\infty} \mathcal{L}(E_{m+\vartheta}, BC^k(\mathbb{R}^+, E_m)) \tag{B.7}$$

and

$$\partial^j R_k g(0) = \delta_k^j g \ , \ 0 \le j \le k \ , \ g \in E_\vartheta \ . \tag{B.8}$$

Proof: Put $\varphi_k(t) := (1/k!)t^k$ and observe that $\partial^\ell \varphi_k = \varphi_{k-\ell}$ for $0 \le \ell \le k$, whereas $\partial^\ell \varphi_k = 0$ for $\ell > k$. Hence

$$\partial^j R_k g(t) = \sum_{\ell=0}^{k\wedge j} \binom{j}{\ell} \varphi_{k-\ell}(t) \Lambda^{j-\ell} e^{t\Lambda} g \ , \ j \in \mathbb{N} \ . \tag{B.9}$$

Thus, given $m \in \mathbb{N}$,

$$\|\partial^j R_k g(t)\|_{E_{m+1+k-j}} \le c_{j,k} \sum_{\ell=0}^{k\wedge j} \|t^{k-\ell} \Lambda^{k-\ell+1} e^{t\Lambda} \Lambda^m g\| \ .$$

Hence, putting $n(k,\ell) := k - \ell + 1$,

$$\Big(\int_0^\infty \|\partial^j R_k g(t)\|^p_{E_{m+1+k-j}} dt \Big) \le c_{j,k} \sum_{\ell=0}^{k\wedge j} \Big(\int_0^\infty \|t^{n(k,\ell)-\vartheta} e^{t\Lambda} \Lambda^m g\|^p \tfrac{dt}{t} \Big)^{1/p} \ . \tag{B.10}$$

By a well known characterization of real interpolation spaces by analytic semigroups (e.g. [43, Theorem 1.14.5]), and by (B.3) and (B.1), we deduce from (B.10) that

$$\begin{aligned} \Big(\int_0^\infty \|\partial^j R_k g(t)\|^p_{E_{m+1+k-j}} dt \Big)^{1/p} &\le c_{j,k} \sum_{\ell=0}^{k\wedge j} \|\Lambda^m g\|_{(E, E_{n(k,\ell)})_{\vartheta/n(k,\ell)}} \\ &\le c'_{j,k} \|\Lambda^m g\|_{E_\vartheta} \le c''_{jk} \|g\|_{E_{m+\vartheta}} \ . \end{aligned}$$

This implies (B.6).

If $j < k$ it follows from (B.9), (B.1), and (B.2) that

$$\|\partial^j R_k g(t)\|_{E_m} \le c_{j,m} \sum_{\ell=0}^{j} t^{k-\ell} \|e^{t\Lambda} \Lambda^m g\|_{E_{j-\ell}} \le c'_{j,m} t^{k-j} e^{-\omega t} \|g\|_{E_m} \ .$$

This shows that

$$[g \mapsto \partial^j R_k g] \in \mathcal{L}(E_m, BC(\mathbb{R}^+, E_m))$$

and that $\partial^j R_k g(0) = 0$ for $0 \le j < k$.

If $j = k$ we deduce from (B.9), (B.1), and (B.2) that

$$\begin{aligned} \|\partial^k R_k g(t) - e^{-t\Lambda} g\|_{E_m} &\le c_{k,m} \sum_{\ell=0}^{k-1} t^{k-\ell} \|e^{t\Lambda} \Lambda^m g\|_{k-\ell} \\ &\le c'_{k,m} t^\vartheta \|\Lambda^m g\|_\vartheta \le c''_{k,m} t^\vartheta \|g\|_{m+\vartheta} \ . \end{aligned}$$

Hence

$$[g \mapsto \partial^k R_k g] \in \mathcal{L}(E_{m+\vartheta}, BC(\mathbb{R}^+, E_m))$$

and $\partial^k R_k g(0) = g$. This proves (B.7) and (B.8). ∎

We put $\mathbb{H}^n := \mathbb{R}^{n-1} \times (0,\infty)$, where $\mathbb{R}^0 := \{0\}$, and denote the generic point of $\mathbb{H}^n$ by $x = (x', t)$. As usual, γ denotes the trace operator, and our spaces of distributions are always spaces of $\mathbb{K}^N$-valued distributions. Finally, BUC denotes the space of bounded and uniformly continuous functions.

THEOREM B.2. *Given $k \in \mathbb{N}$, there exists*

$$R_k \in \mathcal{L}(L_p(\partial\mathsf{H}^n), L_p(\mathsf{H}^n)) \cap \mathcal{L}(BUC(\partial\mathsf{H}^n), BUC(\mathsf{H}^n)) \ , \ 1 < p < \infty \ , \tag{B.11}$$

so that

$$R_k \in \mathcal{L}(B^{s-1/p}_{p,p}(\partial\mathsf{H}^n), H^{s+k}_p(\mathsf{H}^n)) \ , \ s \in [1,\infty) \ , \ 1 < p < \infty \ , \tag{B.12}$$

and

$$R_k \in \mathcal{L}(BUC^s(\partial\mathsf{H}^n), BUC^{s+k}(\mathsf{H}^n)) \ , \ s \in \mathbb{R}^+\backslash\mathbb{N} \ , \tag{B.13}$$

and so that

$$\gamma\partial^j R_k g = \delta^j_k g \ , \ 0 \le j \le k \ , \ g \in B^{1-1/p}_{p,p}(\partial\mathsf{H}^n) \cup BUC^s(\partial\mathsf{H}^n) \ , \ 0 < s < 1 \ . \tag{B.14}$$

Proof: Of course we can assume that $N = 1$. If $n = 1$, all spaces over $\partial\mathsf{H} = \{0\}$ equal $\mathbb{K}$. Then the map defined by $R_k g(t) := (1/k!)t^k e^{-t}g, t \ge 0$, has the desired properties. Hence we can assume that $n \ge 2$.

To simplify the notation we put $F := F(\mathbb{R}^{n-1})$ if $F(\mathbb{R}^{n-1})$ is a space of distributions on $\mathbb{R}^{n-1}$.

Let $p \in (1,\infty)$ be fixed and denote by Λ the L_p-realization of $-(1+\sqrt{-\Delta}) \in \mathcal{L}(\mathcal{D}')$. Then it is known that $-\Lambda \in \mathcal{H}(L_p)$, that $e^{t\Lambda} = e^{-t}P(t)$, where $\{P(t)\,;\, t \ge 0\}$ is the Poisson semigroup on $\mathbb{R}^{n-1}$, and that $E_m(\Lambda) \doteq H^m_p$ (e.g. [43, Section 2.5.3]).

We define R_k by (B.5). Observe that

$$R_k \in \mathcal{L}(L_p, L_p(\mathsf{H}^n))$$

and that R_k is independent of $p \in (1,\infty)$.

Recall that

$$(H^{s_0}_p, H^{s_1}_p)_{\theta,p} \doteq B^{(1-\theta)s_0+\theta s_1}_{p,p} \ , \ s_0, s_1 \in \mathbb{R} \ , \ s_0 \ne s_1 \ , \ 0 < \theta < 1 \ .$$

Moreover it is clear that

$$W^m_p(L_p) \doteq W^m_p(\mathsf{H}^n) \doteq H^m_p(\mathsf{H}^n) \ .$$

Using these facts, (B.12) is for $s \in \mathbb{N}^*$ an easy consequence of Proposition B.1. Since

$$[B^{s_0}_{p,p}, B^{s_1}_{p,p}]_\theta \doteq B^{(1-\theta)s_0+\theta s_1}_{p,p} \ ,$$

and

$$[H^{s_0}_p(\mathsf{H}^n), H^s_p(\mathsf{H}^n)]_\theta \doteq H^{(1-\theta)s_0+\theta s_1}_p(\mathsf{H}^n)$$

for s_0, $s_1 \in \mathbb{R}^+$, $s_0 \ne s_1$, $0 < \theta < 1$, we obtain now (B.12) for $s \in [1,\infty)\backslash\mathbb{N}$ by complex interpolation.

Recall that $R_k g(t) = \varphi_k(t)e^{-t}P(t)g$ and that the Poisson semigroup is a strongly continuous contraction semigroup on BUC of convolutions with kernel $p(x) = ct|x|^{-n}$, where c is chosen so that $\|p(\cdot,1)\|_1 = 1$.

Suppose that $g \in BUC^{m+\mu} \hookrightarrow BUC$ for some $m \in \mathbb{N}$ and $\mu \in (0,1)$. Then $R_k g \in BUC$ and $(x \mapsto R_k g) \in C^\infty(\mathsf{H}^n)$ since $P(\cdot)g$ is harmonic in H^n. Thus, if we can show that there is a constant c so that

$$t^{1-\mu}\|\partial^\alpha R_k(t)g\|_\infty \le c\|g\|_{C^{m+\mu}} \ , \ \alpha \in \mathbb{N}^n \ , \ |\alpha| = m+k+1 \ , \tag{B.15}$$

where $\|\cdot\|_{C^s}$ is the norm in BUC^s, it follows that $R_k \in \mathcal{L}(BUC^{m+\mu}, BUC^{m+k+\mu}(\mathsf{H}^n))$ (e.g. [4, Part I, Appendix 4]).

Observe that $p(x) = t^{-n+1}p(x'/t,1)$ implies $\|p(\cdot,t)\|_1 = \|p(\cdot,1)\|_1 = 1$. Moreover, since the positive homogeneity of p gives

$$\partial^\alpha p(x) = |x|^{-n+1-|\alpha|}\partial^\alpha p(x/|x|) \ , \ \alpha \in \mathbb{N}^n \ ,$$

it follows that

$$|\partial^\alpha p(x)| \le c_\alpha |x|^{-n} t^{1-|\alpha|} = c_\alpha p(x) t^{-|\alpha|} \ , \ \alpha \ne 0 \ .$$

Hence

$$\|\partial^\alpha p(\cdot,t)\|_1 \le c_\alpha t^{-|\alpha|} \quad , \ \alpha \in \mathbb{N}^n \ , \ t > 0 \ . \tag{B.16}$$

Suppose that $\alpha = \beta + \gamma \in \mathbb{N}^n$. Then it follows from (B.16) and Young's inequality that

$$\begin{aligned}\|\partial^\alpha P(t)g\|_\infty &= \|\partial^\alpha [p(\cdot,t/2) * p(\cdot,t/2) * g]\|_\infty \\ &\le c_{\alpha,\beta}\|\partial^\beta p(\cdot,t/2) * \partial^\gamma P(t/2)g\|_\infty \le c'_{\alpha,\beta} t^{-|\beta|}\|\partial^\gamma P(t/2)g\|_\infty \ .\end{aligned} \tag{B.17}$$

Moreover, it is known that

$$t^{1-\mu}\|\partial^\gamma P(t)h\|_\infty \le c_\gamma \|h\|_{C^\mu} \quad , \ |\gamma| = 1 \ , \tag{B.18}$$

and

$$t^{j-\ell-\mu}\|\partial_t^j P(t)h\|_\infty \le c_{j,\ell}\|h\|_{C^{\ell+\mu}} \quad , \ j > \ell \ , \ j,\ell \in \mathbb{N} \ , \tag{B.19}$$

(e.g. [41, §V.4]).

Suppose now that $\alpha \in \mathbb{N}^n$ satisfies $|\alpha| = m + k + 1$ and write $\alpha = \beta + \gamma + je_n$, where $|\gamma| \le m$ and $\beta_n = \gamma_n = 0$. Then it follows from Leibniz' rule that $t^{1-\mu}\partial^\alpha R_k(t)g$ is a linear combination with constant coefficients of terms of the form

$$e^{-t}t^{1-\mu+k-r}\partial_t^{j-\ell}\partial^\beta P(t)\partial^\gamma g \quad , \ 0 \le r \le \ell \wedge k \ , \ 0 \le \ell \le j \le m + k + 1 \ .$$

Thus, thanks to the exponential factor, it suffices to find estimates of the form

$$t^{1-\mu+k-(k\wedge\ell)}\|\partial_t^{j-\ell}\partial^\beta P(t)\partial^\gamma g\|_\infty \le c\|g\|_{C^{m+\mu}} \tag{B.20}$$

in order to verify (B.15).

Suppose first that $j \le k$. Then $|\beta| + |\gamma| = m + k - j + 1 \ge m + 1$. Hence we can assume that $|\gamma| = m$ and $|\beta| \ge 1$. Thus it follows from (B.17) and (B.18) that –choosing $\tilde\beta \le \beta$ with $|\tilde\beta| = 1$–

$$\begin{aligned}t^{1-\mu+k-\ell}\|\partial_t^{j-\ell}\partial^\beta P(t)\partial^\gamma g\|_\infty &\le ct^{1-\mu+k-j-|\beta|+1}\|\partial^{\tilde\beta} P(t/2)\partial^\gamma g\|_\infty \\ &\le c'\|\partial^\gamma g\|_{C^\mu} \le c''\|g\|_{C^{m+\mu}} \ ,\end{aligned}$$

since $j + |\beta| = k + 1$. If $j \ge k + 1$ we can put $\beta = 0$ so that $|\gamma| = m + k + 1 - j$ and $\partial^\gamma g \in BUC^{m-|\gamma|+\mu} = BUC^{j-k-1+\mu}$. Thus, if $\ell \le k$ then $j - \ell > m - |\gamma|$ and we deduce from (B.19) that

$$t^{1-\mu+k-\ell}\|\partial_t^{j-\ell}P(t)\partial^\gamma g\|_\infty \le c\|\partial^\gamma g\|_{C^{m-|\gamma|+\mu}} \le c'\|g\|_{C^{m+\mu}} \ .$$

Finally, if $\ell > k$ we obtain from $BUC^{m-|\gamma|+\mu} \hookrightarrow BUC^{j-\ell-1+\mu}$ and (B.19) that

$$t^{1-\mu}\|\partial_t^{j-\ell}P(t)\partial^\gamma g\|_\infty \le c\|\partial^\gamma g\|_{C^{j-\ell-1+\mu}} \le c'\|g\|_{C^{m+\mu}} \ .$$

Hence (B.20) has been verified in all possible cases. Thus (B.15) is true, so that (B.13) has been proven.

Assertion (B.14) is now an easy consequence of (B.8), the definition of the trace operator, and the strong continuity of the Poisson semigroup on BUC. ∎

Extension operators for boundary values satisfying conditions (B.14) can also be constructed by using the methods of J.-L. Lions [27] (cf. also [5, Lemma 5.1]). However this approach has the disadvantage that the corresponding operators depend on the order of the Bessel potential spaces. Another method, which works in the Hölder space setting, has recently been proposed in [29]. Of course there are many more extension theorems for boundary values (e.g. [26,43]). Our approach has, however, the advantage that it gives — for a fixed $k \in \mathbb{N}$ — a single extension operator which works simultaneously in the class of Hölder and Bessel potential spaces and is independent of the order of these spaces.

We assume now again that Ω is a bounded domain in $\mathbb{R}^n$ of class C^∞ and use the notations of Section 2. We put

$$\mathfrak{B}(\partial\Omega) := \{(b_0, b_1, \dots, b_n) \in C(\partial\Omega, \mathcal{L}(\mathbb{K}^N))^{n+1} \,;\, b_j\nu^j(x) \in \mathcal{GL}(\mathbb{K}^N), x \in \partial\Omega\} \ .$$

Moreover, given $s \in [1,\infty)$ and $p \in (1,\infty)$,

$$\mathfrak{B}_p^s(\partial\Omega) := \mathfrak{B}(\partial\Omega) \cap B_{\hat{p},\hat{p}}^{s-1/\hat{p}}(\partial\Omega, \mathcal{L}(\mathbb{K}^N))^{n+1}$$

and, given $t \in \mathbb{R}^+$,

$$\mathfrak{B}^t(\partial\Omega) := \mathfrak{B}(\partial\Omega) \cap C^t(\partial\Omega, \mathcal{L}(\mathbb{K}^N))^{n+1} \ ,$$

where $\mathfrak{B}(\partial\Omega) \cap X$ is always given the topology of the Banach space X. Observe that $\mathfrak{B}_p^s(\partial\Omega)$ is open in $B_{\hat{p},\hat{p}}^{s-1/\hat{p}}(\partial\Omega, \mathcal{L}(\mathbb{K}^N))^{n+1}$ and that $\mathfrak{B}^t(\partial\Omega)$ is open in $C^t(\partial\Omega, \mathcal{L}(\mathbb{K}^n))^{n+1}$ for $s \in [1,\infty)$ and $p \in (1,\infty)$, and for $t \in \mathbb{R}^+$, respectively.

Given $\beta := (b_0, b_1, \dots, b_n) \in \mathfrak{B}(\partial\Omega)$, we define a boundary operator $\mathcal{B}(\beta)$ by

$$\mathcal{B}(\beta) := \delta(b_j\gamma\partial_j + b_0\gamma) + (1-\delta)\gamma \ .$$

Moreover we put

$$C^t := C^t(\overline{\Omega}, \mathbb{K}^N) \quad , \ t \in \mathbb{R}^+ \ ,$$

and

$$\partial C^t := \prod_{\Gamma\in\mathbf{\Gamma}} \prod_{r=1}^N C^{t-\delta^r(\Gamma)}(\partial\Omega, \mathbb{K}) \quad , \ t \geq 1 \ .$$

Then we can prove the main result of this section.

THEOREM B.3. *There exists a map $\mathcal{R}$ with the following properties:*

- **(i)** $\mathcal{R} \in C^\infty(\mathfrak{B}_p^{s-1}(\partial\Omega), \mathcal{L}(\partial B_p^s, H_p^s)) \quad , \ s \in [2,\infty) \ , \ p \in (1,\infty) \ ;$
- **(ii)** $\mathcal{R} \in C^\infty(\mathfrak{B}^{t-1}(\partial\Omega), \mathcal{L}(\partial C^t, C^t)) \quad , \ t \in (1,\infty)\backslash\mathbb{N} \ ;$
- **(iii)** $\mathcal{B}(\beta)\mathcal{R}(\beta)g = g \quad , \ \gamma\mathcal{R}(\beta)g = (1-\delta)g$ *for* $\beta \in \mathfrak{B}_p^1(\partial\Omega)$ *and* $g \in \partial B_p^2$.

Proof: By means of local coordinates we deduce from Theorem B.2 the existence of linear operators R_k, $k = 0, 1$, so that

$$R_k \in \mathcal{L}(B_{p,p}^{s-1/p}(\partial\Omega, \mathbb{K}^N), H_p^{s+k}) \quad , \ s \in [1,\infty) \ , \ 1 < p < \infty \ , \tag{B.21}$$

and

$$R_k \in \mathcal{L}(C^t(\partial\Omega, \mathbb{K}^N), C^{t+k}) \quad , \ t \in \mathbb{R}^+\backslash\mathbb{N} \ , \tag{B.22}$$

and so that

$$\gamma R_k = \delta_k^0 \quad , \ \partial_\nu R_1 = 1 \ . \tag{B.23}$$

Putting

$$S := R_0 - R_1\partial_\nu R_0 \ ,$$

it follows that

$$S \in \mathcal{L}(B_{p,p}^{s-1/p}(\partial\Omega, \mathbb{K}^N), H_p^s) \quad , \ s \in [2,\infty) \ , \ 1 < p < \infty \ , \tag{B.24}$$

that

$$S \in \mathcal{L}(C^t(\partial\Omega, \mathbb{K}^N), C^t) \quad , \ t \in (1,\infty)\backslash\mathbb{N} \ , \tag{B.25}$$

and that

$$\gamma S = 1 \quad , \ \partial_\nu S = 0 \ . \tag{B.26}$$

Given $\beta \in \mathfrak{B}(\partial\Omega)$, define a tangential differential operator on $\partial\Omega$ by

$$\mathcal{T}(\beta) := b_j\gamma\partial_j - (b_j\nu^j)\partial_\nu \tag{B.27}$$

and observe that

$$\mathcal{B}(\beta) = \delta\{(b_j\nu^j)\partial_\nu + \mathcal{T}(\beta) + b_0\gamma\} + (1-\delta)\gamma \ . \tag{B.28}$$

Moreover, let

$$\mathcal{R}(\beta) := R_1(b_j\nu^j)^{-1}\delta[1 - (\mathcal{T}(\beta)S + b_0)(1-\delta)] + S(1-\delta) \ . \tag{B.29}$$

Observe that

$$[\beta \mapsto (b_j\nu^j)^{-1}] \in C^\infty(\mathfrak{B}^1_p(\partial\Omega) \ , \ B^{s-1/\hat{p}}_{\hat{p},\hat{p}}(\partial\Omega, \mathcal{L}(\mathbb{K}^N))) \tag{B.30}$$

for $s \in [1,\infty)$ and $1 < p < \infty$, whereas

$$[\beta \mapsto (b_j\nu^j)^{-1}] \in C^\infty(\mathfrak{B}^t(\partial\Omega) \ , \ C^t(\partial\Omega, \mathcal{L}(\mathbb{K}^N))) \tag{B.31}$$

if $t \in \mathbb{R}^+$.

Suppose first that $p > n$. Then $\hat{p} = p$ and $B^\sigma_{p,p}(\partial\Omega, \mathbb{K})$ is a topological algebra with respect to pointwise multiplication provided $\sigma > (n-1)/p$, as follows from the fact that $B^\sigma_{p,p}(\mathbb{R}^{n-1}, \mathbb{K})$ has this property for $\sigma > (n-1)/p$ (e.g. [33, Theorem 7.11] or [39, Corollary III.2.1]). Using this fact, assertion (i) follows in this case easily from (B.21), (B.24), (B.27), (B.29), (B.30) and the continuity of the trace operator. If $p \le n < \hat{p}$ the same considerations prove assertion (i) for $s > 1 + n/p$. If $s \le 1 + n/p$ it follows from [33, Theorem 7.10], for example (cf. also [30, Theorem 3.3.1.1] that $B^{s-1-1/\hat{p}}_{\hat{p},\hat{p}}(\partial\Omega, \mathbb{K})$ imbeds in the space of pointwise multipliers for $B^{s-1-1/p}_{p,p}(\partial\Omega, \mathbb{K})$. Now assertion (i) follows in this case by similar arguments as above.

Assertion (ii) is a consequence of (B.22), (B.25), (B.27), (B.29), (B.31), and the fact that $C^t(\partial\Omega, \mathbb{K})$ is a topological algebra with respect to pointwise multiplication for each $t \in \mathbb{R}^+$.

Finally, given $g \in \partial B^2_p$, put $v := \mathcal{R}(\beta)g$ and $w := v - S(1-\delta)g$. Then it follows from (B.23), (B.26), and (B.29) that

$$(b_j\nu^j)\partial_\nu v = \delta[g - (\mathcal{T}(\beta) + b_0\gamma)(v - w)] \ .$$

Hence, thanks to (B.28),

$$\mathcal{B}(\beta)v = \delta[g + (\mathcal{T}(\beta) + b_0\gamma)w] + (1-\delta)\gamma v \ . \tag{B.32}$$

Observe that

$$\gamma v = (1-\delta)g \tag{B.33}$$

by (B.23), (B.26), and (B.29). Hence $\gamma w = \gamma v - \gamma S(1-\delta)g = 0$ by (B.26). Thus $(\mathcal{T}(\beta) + b_0\gamma)w = 0$, due to the fact that $\mathcal{T}(\beta)$ is a tangential differential operator. Now assertion (iii) follows from (B.32) and (B.33). ▮

It should be noted that the dependence of $\mathcal{R}(\beta)$ on b_0 involves only $\delta b_0(1-\delta)$.

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